Duality Theory For Complex And Continuous Nonlinear Programming Problem

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In Partial Fulfilment of the requirements
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By

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CERTIFICATE

This is to certify that the thesis entitled "Duality Theory for Complex and Continuous Nonlinear Programming Problems" by Rajendra Prasad Gupta, for the award of the Degree of Doctor of Philosophy, of the Indian Institute of Technology, Kanpur, is a record of bonafide research work carried out by him under my supervision and guidance for the last three years. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

J. N. Kapur

J.N. KAPUR
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January - 1969

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Rajendra Prasad Gupta

Rajendra Prasad Gupta

TO MY WIFE

CONTENTS

LIST OF SYMBOLS

∇ Gradient operator $\left(\frac{\partial}{\partial X_1} \quad \frac{\partial}{\partial X_2} \text{ ---- } \frac{\partial}{\partial X_n}\right)'$

$\left(\frac{\partial}{\partial X_1(t)}, \frac{\partial}{\partial X_2(t)}, \text{----} \frac{\partial}{\partial X_n(t)}\right)' \left(\frac{\partial U}{\partial X_1}, \frac{\partial U}{\partial X_2}, \text{----} \frac{\partial U}{\partial X_n},\right.$

$\left.\frac{\partial U}{\partial Y_1}, \frac{\partial U}{\partial Y_2}, \cdots \frac{\partial U}{\partial Y_n}\right)'$

U_e : $\left(\frac{\partial U}{\partial e_1}, \frac{\partial U}{\partial e_2}, \cdots \frac{\partial U}{\partial e_n}\right)'$

V_e : $\left(\frac{\partial V}{\partial e_1}, \frac{\partial V}{\partial e_2}, \cdots \frac{\partial V}{\partial e_n}\right)'$

T : Transpose

(') : Prime

$\frac{\pi}{2}$: Vector with each component $\frac{\pi}{2}$

$(\bar{\ })$: Conjugate of

$\Rightarrow$: This implies that

$\sum_i a_i b_i$: Summation over i

$(a,b) = \sum_i \bar{a}_i b_i$: Inner product of two complex vectors

$(a.b) = \sum_i a_i b_i$	:	Inner product of two vectors
Re	:	"Real of"
i	:	Imaginary unit
Arg	:	Argument of
$\mid\ \mid$	:	Absolute value, Modulus of
*	:	Conjugate Transpose
$\exists$	:	There exists
$\{0\}$	:	Null set or Empty set.
$\in$	:	"Belongs to" or "is an element of".

SYNOPSIS

"Duality Theory for Complex and Continuous Nonlinear Programming Problems" a thesis submitted in partial fulfilment for the degree of Doctor of Philosophy by Rajendra Prasad Gupta to the Department of Mathematics, Indian Institute of Technology, Kanpur, January 1969.

The present thesis deals with duality, symmetry and self-duality of complex programs and duality theory of continuous nonlinear programming problems.

Chapter I presents introduction, a brief survey of duality theory in mathematical programming and summary of the work included in the present thesis.

Duality for quadratic programs in real space has been studied by Dennis [20] , Dorn [21] , [22] , [24] and Cottle [14] , [15] . Chapter II is devoted to extending the study of symmetric dual and self-dual quadratic programs to complex space. This chapter is divided into two sections. Section I deals with a pair of symmetric dual quadratic programs in complex space. The duality theorem is proved for this pair by adopting the technique of Dorn [21] and using the duality theorem for linear programming problem in complex space [59] . Self dual linear and formal self-dual quadratic programs are defined in complex space and the self duality theorem is proved for both programs. In section II we have

considered the self dual quadratic program in complex space. This is the generalisation of Dorn's [24] self dual quadratic program. The fundamental theorem in quadratic programming [15] is extended to complex quadratic programming. For this purpose, the properties of real symmetric positive definite and positive semidefinite matrices have been generalised for Hermitian matrices whose forms are positive definite and positive semi-definite.

Chapter III deals with a convex program in complex space. Dorn [25] has proved the duality theorem for convex programs. This duality theorem is generalised for complex convex programs by adopting the technique of Dorn [25] and using duality theorem for linear programming in complex space. Associated to the complex convex program a linear version in complex space is taken, its dual is formed and by the duality theorem [59] the optimal values of these linear programs are shown to be equal. Weak duality theorem, Optimality condition, Complementary slackness theorem and duality theorem are extended for complex convex programs. Duality theorem is proved in two parts. (i) Direct Duality Theorem (ii) Converse Duality Theorem. Weak duality theorem and Direct duality theorem are extended under the assumption that Re $[f(Z)]$ = U(X,Y) is convex function of X and Y such that $X + iY = Z \in C_I$ where C_I is the region in n dimensional complex space defined by primal constraints. The converse duality theorem requires in addition that the inverse of $f'(Z)$ has a derivative.

In chapter IV we have considered the symmetry, duality and self duality of nonlinear programming problems in complex space. Section I

deals with the duality theorem for a symmetric dual pair of programs. This theorem assures that if primal problem is solvable at certain point then dual problem is also solvable at the same point and vice versa. The weak duality theorem is proved under the assumption that Re $[f(Z)] = U(X;Y)$ and Re $[g(W)] = p(\gamma, \delta)$ are convex functions in the region defined by $X + i\,Y = Z$ and $\gamma + i\,\delta = W$ satisfying the primal constraints. Duality theorem requires in addition that the inverse of $f'(Z)$ and $g'(W)$ have derivatives. Section II presents symmetric and self dual nonlinear programs in complex space. The pair of symmetric dual programs is reduced to a self dual program under the assumption that $K(Z,W)$ is skew symmetric and $\beta = \frac{\pi}{2} - \alpha$. By proving self duality theorem it is shown that the optimal value of self dual program is zero.

Chapter V is devoted to the study of duality for continuous nonlinear programming problems. Duality theorems of static nonlinear programming problems are generalized to dynamic nonlinear programming problems. Duality theorem for continuous linear program is proved by Tyndall [85] [86] and Levinson [60] . Under the same two hypotheses [85] the existence of optimal solutions is shown Section I presents the duality theorem for continuous quadratic program under the assumption that the quadratic form is negative semi-definite over the interval. In section II duality theorem for continuous convex program is proved which states that if $Z_o(t)$ is optimal to primal problem then there exists $U(t) = Z_o(t)$, $W(t) = W_o(t)$ such that $(Z_o(t), W_o(t))$ is optimal to dual program.

The work included in the thesis is based on the following papers written by the author.

1. Symmetric Dual Quadratic Program in Complex Space. Accepted in Proceedings of Indian Academy of Sciences.

2. "Self Dual Quadratic Program in Complex Space" Communicated for publication.

3. "Duality Theorem for Convex Program in Complex Space" Accepted for publication in Cahier Du Centre D'Etudes De Recherche Operationnelle.

4. Converse Duality Theorem for Convex Program in Complex Space" Presented to First Annual Convention of Operational Research Society of India, December, 1968.

5. "General Symmetric Dual Programs in Complex Space" Communicated for publication.

6. Symmetric Dual and Self Dual Nonlinear Program in Complex Space" Communicated for publication.

7. Duality Theorem for Continuous Time Quadratic Programming Problems. Reported in Prof. Kapur's Presidential Address to Mathematics Section of India Science Congress, 4th January 1968.

8. Duality Theorem for Continuous Time Convex Programming Problems. Reported in the Address mentioned above.

CHAPTER - I

INTRODUCTION

The purpose of this chapter is to present a brief survey of duality theory in mathematical programming. This survey is not intended to be exhaustive. It is included so as to place the present authors contribution in its proper perspective. This chapter is divided into three sections. Section I presents a brief survey of duality theory in mathematical programming. Section II deals with the duality theory of nonlinear programming in complex space. Section III is concerned with duality theory for continuous nonlinear programming problems.

Mathematical programming is the analysis of problems of the type "to find the maximum (or minimum) of a function when the variables

are subject to equality and inequality constraints". Any problem which seeks to maximize or minimize a numerical function of one or more variables (or functions) when the variables (or functions) can be independent or related in some way through the specification of certain constraints may be referred to as an optimisation problem. For the last two hundred years the classical techniques such as these of differential calculus and calculus of variations were used to solve certain type of optimisation problems. In the last two decades there has been a remarkable growth of interest in a new class of optimization problems, often referred to as programming problems. Unfortunately the ordinary calculus is usually not applicable to programming problems, since their solutions fall on the boundaries of the domain of the variables.

SECTION - I

A Brief Survey of Duality Theory in Mathematical Programming

The mathematical programming problem is a generalisation of the classical problem

$$\text{minimize} \quad f_o(X)$$

$$\text{constrained by} \quad f_i(X) = 0 \qquad 1 \leqslant i \leqslant m$$

which is usually handled by the method of Lagrange multipliers. The Lagrange multiplier approach was first extended to mathematical programming problem by F. John [47] who established necessary and sufficient conditions for a solution.

The mathematical programming problem can be put as follows

$$\text{minimize} \quad f_o(X)$$

$$\text{constrained by} \quad f_i(X) \leqslant 0 \qquad 1 \leqslant i \leqslant m$$

where the $f_i(\ 0 \leq i \leq m)$ are all real valued functions defined on real n space. In particular if all the f_i are linear forms, then the above is a linear program otherwise it is a nonlinear program. The simplest kind of nonlinear program is the quadratic program. When all the functions $f_i(X)$ are convex the above program is called a convex program, in such a program local extrema are always global.

From a mathematical point of view the mathematical programming may be regarded as being divided into two areas, one is primarily analytical and deals with certain questions of duality and consistency while the other is algorithmic and is concerned with computational questions and methods.

Fasbender gave an example in 1846 of a duality between nonlinear programs. He proved that the altitude of any circumscribing equilateral triangle is not greater than the sum of distances for the $X_i(i=1,2,3)$ to any point interior to $\triangle X_1X_2X_3$. Equality between these two quantities is sufficient to prove that we have the circumscribing triangle with maximum altitude and the point X with the minimum distance sum. (Abadie [1])

(1.1.a) <u>Duality Theory for Linear Programming</u>

The duality theory of linear programming was first discussed by Von Neumann in 1947. Later it was developed by Gale, Kuhn, Tucker [51] Bratton [10] , Dantzig [17] , Duffin [26] . The problems considered by Gale, Kuhn and Tucker are the general matrix problems of linear programming which contains the scalar and vector problems as special cases and these problems are related to the theory of zero sum two person games.

<u>Problem 1</u>. To find a maximal matrix D having the property that

$CX \geqslant DY$ for some $X \geqslant 0$, $Y > 0$ such that $AX \leqq BY$.

<u>Problem 2</u>. To find a minimal matrix D having the property that

$B'U \leqslant D'V$ for some $U \geqslant 0$, $V > 0$ such that $A'U \geqslant C'V$.

If the matrix B consists of a single column b and the matrix C onsists of a single row C', then D becomes a scalar δ and Y and V become positive scalars that may be eliminated by dividing throughout by them. The above problems reduce to the following problems.

<u>Problem 1. δ</u>. To find a maximal scalar δ having the property that

$C'X \geqslant \delta$ for some $X \geqslant 0$ such that $AX \leq b$.

<u>Problem 2 . δ</u>. To find a minimal scalar δ having the property that

$b'U \leqslant \delta$ for some $U \geqslant 0$ such that $A'U \geqslant C$

Von Neumann considered the following scalar problems in 1947.

I minimise CX (1.1.1)

subject to $AX + b \geqslant 0$ $\quad X \geqslant 0$ (1.1.2)

II maximise $-bY$ (1.1.3)

subject to $-A'Y + C \geqslant 0$, $\quad Y \geqslant 0$ (1.1.4)

He called I as primal problem and II as dual problem. Since negation and transposition are involutory operations therefore above pair of programs is symmetric.

The major theorems on duality in linear programming are

<u>Weak Duality Theorem</u>. If X and Y are feasible for primal and dual problems respectively then

$$\text{Sup} - bY \leqslant \text{Inf } CX \tag{1.1.5}$$

<u>Duality Theorem</u>. If either program has an optimal solution, then so does the other, and when this is so

$$\text{maximum} \quad -bY = \text{minimum} \quad CX \tag{1.1.6}$$

<u>Complementary Slackness Theorem</u>. $\hat{X}$, $\hat{Y}$ are optimal solutions of primal and dual problems respectively if and only if

$$\hat{Y}\,(\,A\hat{X} + b\,) = 0 \tag{1.1.7}$$

$$\hat{X}\,(-A'\hat{Y} + c) = 0 \tag{1.1.8}$$

<u>Existence Theorem</u>. If primal and dual constraint sets are nonempty, then both problems have optimal solutions.

<u>Unboundedness Theorem</u>. If one of the constraint set is nonempty then either $\sup -bY = +\infty$, or $\inf CX = -\infty$ according as primal constraint set or dual constraint set is empty.

The notion of self duality was first given by Duffin [26] for infinite linear programs. The self dual finite linear programming problem is

<u>Primal Problem</u>.

$$\text{minimise} \quad CX \tag{1.1.9}$$

$$\text{subject to} \quad AX \geqslant -C \quad , \quad X \geqslant 0 \tag{1.1.10}$$

<u>Dual Problem</u>.

$$\text{maximise} \quad -CY \tag{1.1.11}$$

$$\text{subject to} \quad A'Y \leq C \quad , \quad Y \geqslant 0 \tag{1.1.12}$$

<u>Self Duality Theorem</u>. If A is skew symmetric then dual program is identical to primal program and the optimal value is zero.

$$\text{Maximum} \quad -CY = \text{minimum} \quad CX = 0 \tag{1.1.13}$$

Sreedharan [81] gave a short proof of the duality theorem of linear programming by using classical Minkowski Farkas (M.F.) lemma. The M.F. Lemma is $UW \geqslant 0$ for all $UT \geqslant 0$ if and only if there exists $W \geqslant 0$ such that $TW = V$.

(1.1.b) Duality in Quadratic Programming

Since 1959 the notion of duality has been extended to quadratic programming by Dennis [20] and Dorn [21] Dorn [21], [22], [24] studied the duality theorem, symmetric duality theorem and self duality for quadratic programs respectively. The quadratic program considered by Dorn [21] is the following

Primal Program (P.P.)

$$\text{minimise} \quad f(X) = \tfrac{1}{2} X'CX + p'X \tag{1.1.14}$$

$$\text{subject to} \quad AX \geqslant b \tag{1.1.15}$$

$$X \geqslant 0 \tag{1.1.16}$$

where C is a symmetric positive semi-definite nxn matrix and A is mxn matrix, X,p;b are nx1, mx1 column vectors respectively.

The dual problem to (1.1.14), (1.1.15), (1.1.16) is

Dual Program (D.P.)

$$\text{Maximise} \quad g(U,V) = -\tfrac{1}{2} U'CU + b'V \tag{1.1.17}$$

$$\text{subject to } A'V - CU \leqslant p \tag{1.1.18}$$

$$V \geqslant 0 \tag{1.1.19}$$

where U is nx1 vector V is mx1 vector

The duality theorem proved by Dorn is that

(I) If $X = X_o$ is a solution to primal problem then a solution $(U,V) = (X_o,V_o)$ exists to dual problem.

II Conversely if a solution $(U,V) = (U_o,V_o)$ to dual problem exists then a solution which satisfies $CX = CU_o$ to primal problem also exists.

(III) In either case $\text{Max } g(U,V) = \text{Min } f(X)$ (1.2.20)

This theorem is proved by using duality theory of linear programming. The assumption that C is symmetric positive semi-definite makes the function $f(X)$ convex, therefore local minimum is global minimum.

Self-dual quadratic program was formed by Dorn [24] Dorn gave the following definition for a self-dual program. If constraints are added (or substracted) from a program in such a way that the solution (both the optimal value of the objective function and the optimal values of the variables) is unchanged the new program thus constructed is called equivalent to the original program. A program is called self dual if it is equivalent to its dual.

Dorn considered the following program

P minimize
$$f(X_1X_2 \text{———} X_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}X_iX_j + \sum_{i=1}^{n} p_iX_i \quad (1.1.21)$$

subject to
$$\sum_{j=1}^{n} a_{ij}X_j \geqslant p_i \quad (1.1.22)$$

$$X_j \geqslant 0 \quad (1.1.23)$$

and $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}X_iX_j \geqslant 0$ for all X_i equality hold

iff $x_i = 0$ for all i.

First he proved that the program P has a finite solution then the self duality theorem was proved, which states that if a program is equivalent to its dual then it is self dual and the optimal value of the program is zero.

The symmetric duality theorem for quadratic program was proved by Dorn [22] . This theorem is proved by using duality theory of linear programming. A program is said to be symmetric if dual of the dual is the primal program. It is due to this involutory property which Dorn [22] calls symmetry.

In 1963 Cottle [14] formed a more general symmetric dual quadratic program, which includes all the programs considered by Dorn [21] , [22] , [24] as a particular cases. The formal type of self duality has also been discussed by Cottle [14] and he has obtained a composite program from the pair of symmetric dual quadratic programs.

The programs considered by Cottle [14] are

Primal Program (P)

minimise $F(X, Y) = \frac{1}{2} Y'DY + \frac{1}{2}X'CX + p'X$ (1.1.24)

subject to $DY + AX \geqslant - b$ (1.1.25)

$X \geqslant 0$ (1.1.26)

Dual Program (P^*)

maximise $G(U,V) = - \frac{1}{2}V'DV - \frac{1}{2}U'CU - b'V$ (1.1.27)

subject to $- A'V + CU \geqslant - p$ (1.1.28)

$V \geqslant 0$ (1.1.29)

The entries of all matrices and vectors are real numbers. A, C, p and b have the meaning assigned previously. D is mxm symmetric positive semi-definite matrix.

If we write P^* as a minimisation problem and take the dual of P^* we get P as a maximization problem, hence dual of the dual is the

original program. Due to this involutory property programs are called symmetric. The symmetric duality theorem is proved by using the duality theory of linear programming. Since the constraints are linear all the five theorems proved for linear program are also proved for this pair of symmetric dual quadratic program.

The composite program consists of minimizing the difference of primal and dual objective functions over the set of jointly feasible solutions.

The composite program of the pair of symmetric dual quadratic program is

P^{**} minimize $\quad Y'DY + X'CX + p'X + b'Y \qquad (1.1.30)$

subject to $DY + AX \geqslant -b \qquad (1.1.31)$

$CX - A'Y \geqslant -p \qquad (1.1.32)$

$X \geqslant 0 \qquad (1.1.33)$

$Y \geqslant 0 \qquad (1.1.34)$

If we take $M = \begin{pmatrix} D & A \\ -A' & C \end{pmatrix}$

$q = \begin{pmatrix} b \\ p \end{pmatrix}$

$Z = \begin{pmatrix} Y \\ X \end{pmatrix}$

The program P^{**} can be written as follows

P^{***} minimize $\quad Z'MZ + q'Z \qquad (1.1.35)$

subject to $\quad MZ + q \geqslant 0 \qquad (1.1.36)$

$Z \geqslant 0 \qquad (1.1.37)$

If M is positive definite matrix then this program is self-dual considered by Dorn [24] . Cottle has generalised the result of Dorn [24] in the sense that if M is positive semi-definite then the above program is self-dual.

(1.1.c) Duality Theorem for Nonlinear Programming Problem.

Duality theorem for nonlinear programming has been studied by Dorn [25] , Wolfe [89] , Hanson [40] , Mangasarian [61] , [62] , [63] , Huard [46] , and symmetric dual nonlinear programs are formed by Dantzig, Eisenberg and Cottle [18] , Mond [66] , Mehndiratta [64] , Cottle and Mond [67] and Hanson [41] formed the self dual nonlinear programs. Sinha [82] and Mehndiratta [65] proved the duality theorem and symmetric duality theorem for nonlinear programming problems respectively. Levinson [59] formed the linear programming problem in complex space and proved the duality theorem. Later the duality theorem for quadratic program in complex space was proved by Hanson and Mond [42] . Duality theorem for continuous linear programming problem was proved by Tyndall [85] , [86] and for continuous quadratic and continuous convex program by Hanson and Mond [43] , [44] , and Gupta. The Duality theorem for the class of continuous linear and continuous convex programming problems have been studied by Levinson [60] and Hanson [45] respectively. Duality in variational problems has been proved by Mond and Hanson [68], [69]. Duality theorem for Geometric programming has been proved by Duffin and Peterson [28] .

Dorn [25] has considered the following problem.

Primal Program (P_1)

minimise	$f(X)$	(1.1.38)
subject to	$AX \geqslant b$	(1.1.39)
	$X \geqslant 0$	(1.1.40)

where $f(X)$ is a differentiable convex scalar function of the vector X. A is $m \times n$ matrix b and X are vectors.

The dual problem to (P_1) is

<u>Dual Program</u> (D_1)

Maximize $\quad g(U,V) = f(U) - U' \nabla f(U) + b'V \qquad (1.1.41)$

subject to $\quad A'V - \nabla f(U) \leqslant 0 \qquad (1.1.42)$

$V \geqslant 0 \qquad (1.1.43)$

The duality theorem, proved for this pair, states that

(a) If there exists a vector X_0 which minimizes $f(X)$ in primal problem (P_1), then there also exist vectors $U = X_0$, $V = V_0$ which maximizes $g(U,V)$ in dual problem (D_1).

(b) If U_0 and V_0 are vectors which maximizes $g(U,V)$ in problem D_1, then $X = U_0$ minimises $f(X)$ in problem P_1.

In either case Minimum $f(X)$ = Maximum $g(U,V)$.

The proof of the theorem is based on the duality theory of linear programming. The first part of the theorem is proved under the weaker assumption that $f(X)$ is convex. The proof of second part of the theorem requires in addition that the inverse gradient of the function f has a derivative.

Wolfe [89] formulated a dual to the problem of minimizing a differentiable convex function under nonlinear concave constraints. This dual problem reduces to the case of linear constraints.

<u>Primal Problem</u> (P_2)

minimize $\quad f(X) \qquad (1.1.44)$

subject to $\quad g_i(X) \geqslant 0 \qquad (1.1.45)$

$i = 1,2 \text{ ------ } m$

<u>Dual Problem</u> (D_2)

$$\text{maximise} \qquad f(X) - \sum_{i=1}^{m} u_i g_i(X) \tag{1.1.46}$$

$$\text{subject to} \quad \nabla f(X) = \sum_{i=1}^{m} u_i \nabla g_i(X) \tag{1.1.47}$$

$$U \geqslant 0 \tag{1.1.48}$$

where $f(X)$ is a convex differentiable function X, $g_i(X)$ for each $i = 1,2$ -----m is a concave differentiable function of X.

He proved the following three theorems.

(I) <u>Weak Duality Theorem</u>. Let V be the infimum of $f(X)$ under the constraints of P_2 and v be the supremum of dual under the constraints of D_2, then $v \leqslant V$.

(II) <u>Duality Theorem</u>. If X_o is the solution of primal problem then $\exists$ U^o such that (X^o, U^o) is solution to the dual problem and the extrema are equal.

(III) <u>Unboundedness Theorem</u>. If the primal problem has only linear constraints and these are inconsistent and the constraints of the dual problem are consistent, then the supremum of objective function of the dual problem is $+\infty$.

The basic difference between the nonlinear duality results and those of linear programming is that in the nonlinear case the function f of primal problem appears not only in the constraints of the dual as expected but remains involved in its objective function as well. Due to this fact the usefulness of the dual problem for computational purposes diminishes.

Hanson [40] has given a theorem similar to that of Wolfe's and its converse. He has assumed the existence and the differentiability of the inverse of the gradient of the Lagrangian function which does not appear in the statement of the problem.

Huard [46] has proved the duality theorem and its converse. The converse duality theorem is proved under the assumption that the matrix $\frac{\partial^2 \theta}{\partial x^2}$ is nonsingular at optimal solution where θ is the Lagrangian function.

Mangasarian [61] has only proved the converse duality theorem. He has given the conditions under which it holds more explicitely. The converse duality theorem has been extended for pseudo convex objective function and the quasi convex constraints on a convex set in [62] . The direct duality theorem is not amenable to such an extension. The extended [illegible] theorem is as follows:

Let $\theta(X)$; $g_1(X), g_2(X)$------ $g_n(X)$ be differentiable functions on E^n. Let C be a convex set in E^n and $\theta(X)$ be pseudo convex on C and $g_1(X)$------$g_n(X)$ be quasi convex on C. If there exists an $X^o \in C$ and $Y^o \in E^n$ satisfying the Kuhn Tucker differentiable conditions [54] , namely

$$\nabla_X \theta(X^o) + \nabla_X \sum_{i=1}^{n} Y_i^o g_i(X^o) = 0 \tag{1.1.49}$$

$$\sum_{i=1}^{n} Y_i^o g_i(X^o) = 0 \tag{1.1.50}$$

$$g_i(X^o) \leq 0 \quad i = 1,2 \text{ ------ } n \tag{1.1.51}$$

$$Y_i^0 \geqslant 0 \quad i = 1,2 \text{ ------ } n \tag{1.1.52}$$

then $$\theta(X^0) = \min_{X \in C} \left\{ \theta(X) \;\middle|\; g_i(X) \leqslant 0, \quad i = 1,2 \text{ ------ } n \right\} \tag{1.1.53}$$

Mond [66] proved the symmetric dual theorem for nonlinear programming using the duality theorem for linear programming. The pair of symmetric dual programs considered by Mond is

Primal Program (P)

$$\text{minimize} \qquad H(X,Y) = f(X,Y) - Y' \nabla_Y f(X,Y) \tag{1.1.54}$$

$$\text{subject to} \quad -\nabla_Y f(X,Y) \geqslant 0 \tag{1.1.55}$$

$$X \geqslant 0 \tag{1.1.56}$$

Dual Program (D)

$$\text{maximize} \qquad G(X,Y) = f(X,Y) - X' \nabla_X f(X,Y) \tag{1.1.57}$$

$$\text{subject to} \quad -\nabla_X f(X,Y) \leqslant 0 \tag{1.1.58}$$

$$Y \geqslant 0 \tag{1.1.59}$$

where $f(X,Y)$ has a continuous first partial derivatives and it is convex in X for fixed Y, and concave in Y for fixed X beside these $f(X,Y)$ is also satisfying 4th and 5th assumptions given in [66] .

Mehndiratta [64] has proved weak duality theorem. Duality Theorem, Unboundedness Theorem, for the following pair of symmetric dual programs

Primal Program (P)

$$\text{minimize} \qquad F(X,Y) = Y' \nabla g(Y) - g(Y) + f(X) \tag{1.1.60}$$

$$\text{subject to} \quad \nabla g(Y) + AX \geqslant -b \tag{1.1.61}$$

$$X \geqslant 0 \tag{1.1.62}$$

Dual Program (D)

subject to $\quad -A'V + \nabla f(U) \geqslant 0$ (1.1.64)

$V \geqslant 0$ (1.1.65)

where $f(X)$ and $g(Y)$ are convex scalar functions possessing continuous and strictly locally convex first partial derivatives. The duality theorem is proved by using duality theory of linear programming.

Dantzig, Eisenberg and Cottle [18] also proved weak duality theorem and duality theorem for a pair of symmetric dual programs.

<u>Primal Program</u> (P)

minimize $\quad \xi(X,Y) = K(X,Y) - Y^T K_2(X,Y)$ (1.1.66)

subject to $\quad K_2(X,Y) \leq 0$ (1.1.67)

$X \in R^n_+ \ , \quad Y \in R^m_+$ (1.1.68)

<u>Dual Program</u> (D)

maximize $\quad \eta(X,Y) = K(X,Y) - X^T K_1(X,Y)$ (1.1.69)

subject to $\quad K_1(X,Y) \geqslant 0$ (1.1.70)

$X \in R^n_+ , \quad Y \in R^m_+$ (1.1.71)

The duality theorem is proved by using Kuhn Tucker optimality conditions. The function $K(X,Y)$ is having the four properties given in [18] .

Mond and Cottle [67] proved the self duality theorem for mathematical programming. They have considered the same pair of programs as considered in [18] . The theorem states that if K is differentiable and skew symmetric ($K(X,Y) = -K(Y,X)$) then P and D are formally identical, if (X_o,Y_o) is a joint optimal solution then so is (Y_o,X_o) and $\xi(X_o,Y_o) = K(X_o,Y_o) = 0$.

Self duality theorem for nonlinear programming problem is proved by Hanson [41] . He has considered the following programs.

Primal Program (P)

minimize $X'f(X)$ (1.1.72)

subject to $f_i(X) \geqslant 0 \qquad i = 1,2 \text{ ------ } m$ (1.1.73)

$X \geqslant 0$ (1.1.74)

Dual Program (D)

maximize $g(X,V) = (X-V)' f(X)$ (1.1.75)

subject to $\nabla \left[(X-V)' f(X)\right] \geqslant 0$ (1.1.76)

$V \geqslant 0$ (1.1.77)

where $X'f(X)$ is differentiable convex function while $f_i(X)$ are differentiable concave functions. The statement of the theorem is: If the primal program considered above has a finite solution, it is self dual, its optimal value is zero and its optimal solution lies at an extreme point of the constraint set.

Duality theorem for variational problems has been proved by Mond and Hanson [68] . They considered the following problem.

Primal Program (P)

minimize $\int_{t_o}^{t_1} f(t,X,X')dt$ (1.1.78)

subject to $Q(t,X,X') \geqslant 0$ (1.1.79)

$X(t_o) = X_o$

$X(t_1) = X_1$ (1.1.80)

Dual Program (D)

maximize $\int_{t_o}^{t_1} \left[f(t,X,X') - \lambda(t)Q(t,X,X')\right] dt$ (1.1.81)

subject to $f_X(t,X,X') - \lambda(t)Q_X(t,X,X') = \frac{d}{dt}\left[f_{X'}(t,X,X')\right.$

$$\left. - \lambda(t)Q_{X'}(t,X,X')\right] \qquad (1.1.82)$$

$$X(t_o) = X_o$$

$$X(t_1) = X_1$$

$$\lambda(t) \geq 0 \qquad (1.1.83)$$

Both fixed end point and free boundary value problems are considered. They have proved the following three theorems.

Theorem I. If f is convex and Q concave in X and X' then the infimum of P is greater than or equal to the supremum of D.

Theorem II (Duality Theorem) If function $X^*(t)$ minimises the primal problem P, then there exists a $\lambda^*(t)$ such that $(X^*(t), \lambda^*(t))$ maximises the dual problem D and the extreme values of P and D are equal.

Theorem III (Converse Duality Theorem) If $(X^*(t), \lambda^*(t))$ is a maximising function for D such that $(X^*(t), \lambda^*(t))$ is piece-wise smooth in $t_o \leq t \leq t_1$ and such that

$\mu(t)G_X - \frac{d}{dt}(\mu(t)G_{X'}) + \frac{d^2}{dt^2}(\mu(t)G_{X''}) = 0$ only has a solution

$\mu(t) = 0$ $t_o \leq t \leq t_1$, then $X^*(t)$ is a minimizing solution of P and the extreme values of P and D are equal

(where $G(t,X,X',\lambda,\lambda') = \left[f_X - \lambda Q_X - (f_{X't} - \lambda Q_{X't}) + \lambda' Q_{X'} - (f_{X'} - \lambda Q_{X'})_X X'\right.$

$$\left. - (f_{X'} - \lambda Q_{X'})_{X'} X''\right])$$

Mond and Hanson [69] has also studied the symmetric duality theorem and self duality for variational problems. The pair of programs considered by them is

<u>Primal Program</u> (P)

$$\text{minimize} \quad \int_{t_o}^{t_1} \Big\{ f(t,X,X',Y,Y') - Y(t) f_Y(t,X,X',Y,Y') + Y(t) \frac{d}{dt} \Big[f_{Y'}(t,X,X',Y,Y') \Big] \Big\} \, dt \qquad (1.1.84)$$

$$\text{subject to} \quad \frac{d}{dt} \; f_{Y'}(t,X,X',Y,Y') \geqslant f_Y(t,X,X',Y,Y') \qquad (1.1.85)$$

$$X(t_o)=X_o, \quad X(t_1)=X_1, \quad Y(t_o)=Y_o, \quad Y(t_1)=Y_1 \qquad (1.1.86)$$

$$X(t) \geqslant 0 \qquad (1.1.87)$$

<u>Dual Program</u> (D)

$$\text{maximize} \quad \int_{t_o}^{t_1} \Big\{ f(t,X,X',Y,Y') - X(t) f_X(t,X,X',Y,Y') + X(t) \frac{d}{dt} \Big[f_{X'}(t,X,X',Y,Y') \Big] \Big\} \, dt \qquad (1.1.88)$$

$$\text{subject to} \quad \frac{d}{dt} \; f_{X'}(t,X,X',Y,Y') \leqslant f_X(t,X,X',Y,Y') \qquad (1.1.89)$$

$$X(t_o)=X_o, \; X(t_1)=X_1, \; Y(t_o)=Y_o, \; Y(t_1)=Y_1 \qquad (1.1.90)$$

$$Y(t) \geqslant 0 \qquad (1.1.91)$$

SECTION - II

<u>Nonlinear Programming Problems In Complex Space</u>

Levinson [59] has proved theorems on linear programming and minimax theorem for matrices with complex elements generalizing the familiar real cases. He has considered the following linear programming problem in complex space.

<u>Primal Program</u> (P.P)

Find a Z if there is any such that

$$\left|\operatorname{Arg} Z\right| \leq \alpha \tag{1.2.1}$$

$$\left|\operatorname{Arg} (AZ - b)\right| \leq \beta \tag{1.2.2}$$

and $$\operatorname{Re} (c^*Z) \text{ is minimized} \tag{1.2.3}$$

<u>Dual Program</u> (D.P)

Find a W if there is any such that

$$\left|\operatorname{Arg} W\right| \leq \frac{\pi}{2} - \beta \tag{1.2.4}$$

$$\left|\operatorname{Arg} (-A^*W + c)\right| \leq \frac{\pi}{2} - \alpha \tag{1.2.5}$$

and $$\operatorname{Re} (b^*W) \text{ is maximized} \tag{1.2.6}$$

A is $m\times n$ matrix b,c are vectors whose entires are from the field of complex numbers, α, β are real vectors with $0 \leq \alpha \leq \frac{\pi}{2}$, $0 \leq \beta \leq \frac{\pi}{2}$.

Duality theorem proved for the above pair of programs states that if there is a feasible Z satisfying primal constraints and if there is an $M < \infty$ such that

$$\operatorname{Re} (c^*Z) \geq -M > -\infty$$

for all feasible Z, then there exists an optimal feasible vector ζ for the linear programming problem. Moreover the dual problem also has an optimal feasible W and

$$\operatorname{Re} (c^*\zeta) = \operatorname{Re} (b^*W) \tag{1.2.7}$$

Hanson and Mond [42] have proved duality theorem for quadratic program in complex space. They have adopted the technique of Dorn [21]

and used the duality theory of linear programming in complex space.

The quadratic program in complex space is as follows:

Primal Program (P.P)

minimize $\operatorname{Re}\,(p^*Z + \tfrac{1}{2}Z^*\,CZ)$ (1.2.8)

subject to $|\operatorname{Arg}(AZ-b)| \leq \beta$ (1.2.9)

$|\operatorname{Arg}\,Z\,| \leq \alpha$ (1.2.10)

Dual Program (D.P)

maximize $\operatorname{Re}\left[b^*W - \tfrac{1}{2}U^*CU\right]$ (1.2.11)

subject to $|\operatorname{Arg}\,(-A^*W + CU + p)| \leq \frac{\pi}{2} - \alpha$ (1.2.12)

$|\operatorname{Arg}\,W\,| \leq \frac{\pi}{2} - \beta$ (1.2.13)

C is nxn Hermitian matrix such that the corresponding Hermitian form is positive semi-definite.

Chapter II of the present thesis consists of two sections. In section I we have presented two quadratic programs in complex space, which are dual and symmetric. Some duality theorems are proved for this pair of programs. The proof of duality theorem is based on the duality theory of linear programming in complex space.

The symmetric dual quadratic programming problem in complex space is:

Primal Program (P.P)

minimize $F(Z,W) = \operatorname{Re}\left[\tfrac{1}{2}W^*DW + \tfrac{1}{2}Z^*CZ + p^*Z\right]$ (1.2.14)

subject to $|\operatorname{Arg}\,(DW + AZ - b)| \leq \beta$ (1.2.15)

$|\operatorname{Arg}\,Z| \leq \alpha$ (1.2.16)

Dual Program (D.P)

$$\text{maximize} \quad G(\lambda, \mu) = \operatorname{Re}\left[-\tfrac{1}{2}\mu^* D\mu - \tfrac{1}{2}\lambda^* C\lambda + b^*\mu\right] \tag{1.2.17}$$

$$\text{subject to} \quad \left|\operatorname{Arg}(-A^*\mu + C\lambda + p)\right| \leq \frac{\pi}{2} - \alpha \tag{1.2.18}$$

$$\left|\operatorname{Arg}\mu\right| \leq \frac{\pi}{2} - \beta \tag{1.2.19}$$

Where A is mxn matrix b,p are mx1 and nx1 vectors whose entries are from the field of complex numbers. C and D are Hermitian positive semi-definite nxn and mxm matrices respectively Z, λ, W, μ are n, m component complex vectors.

Here and throughout $\frac{\pi}{2}$ denotes a vector of appropriate dimension with $\frac{\pi}{2}$ in each entry. If C and D are not Hermitian then as any square matrix with complex entries can be written as a sum of Hermitian and skew Hermitian matrices we have

$$C = \frac{C + C^*}{2} + \frac{C - C^*}{2}$$

($\frac{C + C^*}{2}$ is Hermitian and $\frac{C - C^*}{2}$ is skew Hermitian)

$$\text{and} \quad Z^*CZ = Z^*\left(\frac{C + C^*}{2}\right)Z + Z^*\left(\frac{C - C^*}{2}\right)Z \tag{1.2.20}$$

Since in (1.2.20) first part on the right hand side is Hermitian form and second is skew Hermitian form therefore it follows that

$\operatorname{Re}(Z^*CZ) = \operatorname{Re} Z^*\left(\frac{C + C^*}{2}\right)Z$ and the corresponding positive semi-definiteness requirement would be that $\operatorname{Re}(Z^*CZ) \geq 0$ for all Z.

Self-dual linear program in complex space and formal type of self-dual quadratic program in complex space are defined and self duality theorem is proved for each program sperately. The composite program is obtained from the symmetric dual quadratic program in complex space.

Self dual linear program in complex space is

<u>Primal Program</u> (P.P)

$$\text{minimize} \quad \mathrm{Re}\,(b^*Z) \tag{1.2.21}$$

$$\text{subject to} \quad \left|\mathrm{Arg}\,(AZ + b)\right| \le \frac{\pi}{2} - \alpha \tag{1.2.22}$$

$$\left|\mathrm{Arg}\, Z\right| \le \alpha \tag{1.2.23}$$

<u>Dual Program</u> (D.P)

$$\text{maximize} \quad \mathrm{Re}\,(- b^*W) \tag{1.2.24}$$

$$\text{subject to} \quad \left|\mathrm{Arg}\,(-A^*W + b)\right| \le \frac{\pi}{2} - \alpha \tag{1.2.25}$$

$$\left|\mathrm{Arg}\, W\right| \le \alpha \tag{1.2.26}$$

Where A is skew Hermitian matrix, b is n component complex vector, α is n component real vector with $0 \le \alpha \le \frac{\pi}{2}$.

Formal self-dual quadratic program in complex space is obtained from the pair of symmetric dual quadratic program in complex space by replacing D by C and β by α .

<u>Primal Program</u> (P.P)

$$\text{minimize} \quad \phi\,(Z,W) = \mathrm{Re}\left[\tfrac{1}{2}W^*CW + \tfrac{1}{2}Z^*CZ + p^*Z\right] \tag{1.2.27}$$

$$\text{subject to} \quad \left|\mathrm{Arg}\,(CW + AZ + p)\right| \le \frac{\pi}{2} - \alpha \tag{1.2.28}$$

$$\left|\mathrm{Arg}\, Z\right| \le \alpha \tag{1.2.29}$$

<u>Dual Program</u> (D.P)

$$\text{maximize} \quad \psi\,(\lambda, \mu) = \mathrm{Re}\left[-\tfrac{1}{2}\mu^* C\mu - \tfrac{1}{2}\lambda^* C\lambda - p^*\mu\right] \tag{1.2.30}$$

$$\text{subject to} \quad \left|\mathrm{Arg}\,(-A^*\mu + C\lambda + p)\right| \le \frac{\pi}{2} - \alpha \tag{1.2.31}$$

$$\left|\mathrm{Arg}\, \mu\right| \le \alpha \tag{1.2.32}$$

The notion of self duality for convex quadratic program was first given by Dorn [24] . It was later extended to convex nonlinear program by Hanson [41] . These programs require the the concept of equivalent program given by Dorn [24] . A program is called self dual if it is equivalent to its dual.

In section II of Chapter II, self dual quadratic program in complex space is given and the self duality theorem is proved. The dundamental theorem for quadratic program given by Cottle [15] has also been extended to quadratic program in complex space and the significance of quadratic program in complex space is shown.

The self dual quadratic program in complex space is

<u>Primal Program</u> (P.P)

$$\text{minimize} \quad f(Z) = \operatorname{Re}\left[Z^*CZ + p^*Z\right] \tag{1.2.33}$$

$$\text{subject to} \quad \left|\operatorname{Arg}(CZ + p)\right| \leq \frac{\pi}{2} - \alpha \tag{1.2.34}$$

$$\left|\operatorname{Arg} Z\right| \leq \alpha \tag{1.2.35}$$

<u>Dual Program</u> (D.P)

$$\text{maximize} \quad g(\lambda, \mu) = \operatorname{Re}\left[-p^*\mu - \lambda^* C \lambda\right] \tag{1.2.36}$$

$$\text{subject to} \quad \left|\operatorname{Arg}(-C^*\mu + 2C\lambda + p)\right| \leq \frac{\pi}{2} - \alpha \tag{1.2.37}$$

$$\left|\operatorname{Arg} \mu\right| \leq \alpha \tag{1.2.38}$$

where C is an nxn Hermitian positive definite matrix p,Z are n component complex vectors and α is n component real vector with $0 \leq \alpha \leq \frac{\pi}{2}$.

In Chapter III of the present thesis we have proved duality theory for convex program in complex space. This theorem is proved

by adopting the technique of Dorn [25] and using the duality theory for linear programming in complex space [59] .

The complex convex programming problem is

<u>Primal Program</u> (P.P)

$$\text{minimize} \quad F(Z) = \text{Re}\,\left[f(Z)\right] \tag{1.2.39}$$

$$\text{subject to} \quad \left|\text{Arg}\,(AZ - b)\right| \leq \beta \tag{1.2.40}$$

$$\left|\text{Arg}\,Z\right| \leq \alpha \tag{1.2.41}$$

<u>Dual Program</u> (D.P)

$$\text{maximize} \quad G(\mu, W) = \text{Re}\left[f(\mu) - \left\{\left[\overline{f'(\mu)}\right], \mu\right\} + b^{*}W\right] \tag{1.2.42}$$

$$\text{subject to} \quad \left|\text{Arg}\,(-A^{*}W + [\overline{f'(\mu)}])\right| \leq \frac{\pi}{2} - \alpha \tag{1.2.43}$$

$$\left|\text{Arg}\,W\right| \leq \frac{\pi}{2} - \beta \tag{1.2.44}$$

A, b, α, β are as defined previously, $f(Z)$ is a differentiable function of complex variable Z and $\text{Re}\,[f(Z)] = U(X,Y)$ is convex function of X and Y such that $X + iY = Z \in C_I$ where C_I is the region defined by Z satisfying primal constraints.

The associated linear problem is

$$\text{minimize} \quad \text{Re}\,\left[-f(Z_o) + ([\overline{f'(Z_o)}], Z)\right] \tag{1.2.45}$$

$$\text{subject to} \quad \left|\text{Arg}\,(AZ - b)\right| \leq \beta \tag{1.2.46}$$

$$\left|\text{Arg}\,Z\right| \leq \alpha \tag{1.2.47}$$

The constraints of primal problem and (1.2.45) (1.2.46) (1.2.47) are same. If Z_o is an optimal solution to primal problem then it is also optimal to linear problem (1.2.45) (1.2.46) (1.2.47).

Further the dual of the associated linear problem is

$$\text{maximize} \quad \mathrm{Re}\left[-f(Z_o) + b^* W\right] \tag{1.2.48}$$

$$\text{subject to} \quad \left|\mathrm{Arg}\left(-A^*W + \overline{[f'(Z_o)]}\right)\right| \le \frac{\pi}{2} - \alpha \tag{1.2.49}$$

$$\left|\mathrm{Arg}\, W\right| \le \frac{\pi}{2} - \beta \tag{1.2.50}$$

Levinson's duality theorem [[59], Theorem (2.2)] states that if a solution Z_o to primal problem exists then a solution to the dual problem exists and

$$\mathrm{Re}\left[-f(Z_o) + \left(\overline{[f'(Z_o)]}\ , Z_o\right)\right] = \mathrm{Re}\left[-f(Z_o) + b^*W_o\right] \tag{1.2.51}$$

using this and the weaker assumption that

Re $(f(Z)) = U(X,Y)$ is convex function of X and Y, direct duality theorem is proved.

For proving the converse duality theorem the associated linear problem is

$$\text{maximize} \quad H(W, \xi) = \mathrm{Re}\left[f(\mu_0) - \xi^T \mu_0 + b^*W\right] \tag{1.2.52}$$

$$\text{subject to} \quad \left|\mathrm{Arg}\left(-A^*W + \overline{\xi}\right)\right| \le \frac{\pi}{2} - \alpha \tag{1.2.53}$$

$$\left|\mathrm{Arg}\, W\right| \le \frac{\pi}{2} - \beta \tag{1.2.54}$$

If $\xi = f'(\mu)$ then the constraints of dual problem and (1.2.52) (1.2.53) (1.2.54) are same. Further if (μ_0, W_o, ξ_0) is a maximizing solution to dual problem then (W_o, ξ_0) maximizes the above problem. The duality theory given by Levinson [59] assures the existence of optimal solution to its dual, which is

$$\text{minimize} \quad F(Z) = \mathrm{Re}\left[f(\mu_0)\right] \tag{1.2.55}$$

$$\text{subject to} \quad \left|\mathrm{Arg}\,(AZ-b)\right| \le \beta \tag{1.2.56}$$

$$\left|\operatorname{Arg} Z\right| \leq \alpha \qquad (1.2.57)$$

$$Z = \mu_0 \qquad (1.2.58)$$

and

$$\operatorname{Re}\left[f(\mu_0)\right] = \operatorname{Re}\left[f(\mu_0) - \mathcal{J}_0^T \mu_0 + b^* W_o\right] \qquad (1.2.59)$$

This proves the converse duality theorem.

In Chapter IV of the present thesis we have proved the symmetric duality theorem for nonlinear program in complex space by adopting the technique Dorn [25] and Mond [66] and using the duality theory of linear programming in complex space [59]. Section I deals with the following pair of symmetric dual programs in complex space.

<u>Primal Program</u> (P.P)

$$\text{minimize} \quad F(Z,W) = \operatorname{Re}\left\{ \left[g'(W)\right]^T W - g(W) + f(Z)\right\} \qquad (1.2.60)$$

$$\text{subject to} \quad \left|\operatorname{Arg}\left(AZ + \overline{\left[g'(W)\right]} - b\right)\right| \leq \beta \qquad (1.2.61)$$

$$\left|\operatorname{Arg} Z\right| \leq \alpha \qquad (1.2.62)$$

<u>Dual Program</u> (D.P)

$$\text{maximize} \quad G(\lambda, \mu) = \operatorname{Re}\left[-g(\mu) + f(\lambda) - \left[f'(\lambda)\right]^T \lambda + b^* \mu\right] \qquad (1.2.63)$$

$$\text{subject to} \quad \left|\operatorname{Arg}\left(-A^* \mu + \overline{\left[f'(\lambda)\right]}\right)\right| \leq \frac{\pi}{2} - \alpha \qquad (1.2.64)$$

$$\left|\operatorname{Arg} \mu\right| \leq \frac{\pi}{2} - \beta \qquad (1.2.65)$$

In section II we have presented the following pair of symmetric dual programs in complex space.

<u>Primal Program</u> (P.P)

$$\text{minimize} \quad F(Z,W) = \operatorname{Re}\left[K(Z,W) - \left\{K_2(Z,W)\right\}^T W\right] \qquad (1.2.66)$$

subject to $\left| \operatorname{Arg} \left(\overline{-K_2(Z,W)} \right) \right| \leq \beta$

$$\left| \operatorname{Arg} Z \right| \leq \beta$$

Dual Program (D.P)

Maximize $G(Z,W) = \operatorname{Re} \left[K(Z,W) - \left[K_1(Z,W) \right]^T Z \right]$

subject to $\left| \operatorname{Arg} \overline{K_1(Z,W)} \right| \leq \frac{\pi}{2} - \alpha$

$$\left| \operatorname{Arg} W \right| \leq \frac{\pi}{2} - \beta$$

This pair of programs has been reduced to self dual program under the assumption that $K(Z,W)$ is skew symmetric function and the self duality theorem is proved.

SECTION - III

Duality Theorem for Continuous Nonlinear Programming Problems

R. Bellman has discussed a class of continuous linear programming problems in his book Dynamic Programming [7] . These are called 'bottleneck problems'. It is shown that to each (primal) program can be associated a dual program. These problems were also discussed by P. Wolfe [90] , Lehman [57] Koopman .

Tyndall [85], [86] has treated rigorously a continuous linear programming problem. Under suitable hypotheses he has proved an analogue of the fundamental duality theorem of linear programming which is valid for this class of continuous linear programming problems. He has considered the following problems

Primal Program (P.P)

Maximise $\int_0^T a(t)Z(t)\,dt$ (1.3.1)

subject to $Z(t)$ is bounded and measurable function (1.3.2)

$$Z(t) \geq 0 \qquad 0 \leq t \leq T \tag{1.3.3}$$

$$BZ(t) \leq c(t) + \int_0^t CZ(s)ds \qquad 0 \leq t \leq T \tag{1.3.4}$$

<u>Dual Program</u> (D.P)

$$\text{minimise} \quad \int_0^T W(t)c(t)dt \tag{1.3.5}$$

subject to $W(t)$ is bounded and measurable function (1.3.6)

$$W(t) \geq 0 \qquad 0 \leq t \leq T \tag{1.3.7}$$

$$B'W(t) \geq a(t) + \int_t^T C'W(s)ds \qquad 0 \leq t \leq T \tag{1.3.8}$$

where $a(t)$, $c(t)$ are bounded and measurable functions B and C are constant matrices.

He has proved that under the following two hypotheses

(i) $\left\{ X \in E^N,\ BX \leq 0 \ \text{ and } \ X \geq 0 \right\} = \left\{ 0 \right\}$

(ii) B, C, $c(t)$ have nonnegative components for $0 \leq t \leq T$.

there exists optimal solutions $\overline{Z}$ and $\overline{W}$ to the primal and dual programs respectively and any two feasible functions Z, W are optimal if and only if $\int_0^T Z(t)\, a(t)dt = \int_0^T W(t)c(t)dt$. Beside this theorem he has also proved weak Duality Theorem, Optimality Condition, Equilibrium Conditions and neither hypotheses alone is sufficient to guarentee the existence of optimising solution.

N Levinson [60] generalized the problem by taking the matrices B and C to be time dependent and assuming $a(t)$ $c(t)$ $B(t)$, $K(t)$ to be only piece wise continuous. He has considered the following pair of programs.

Primal Program (P.P)

$$\text{maximise} \quad \int_0^T Z(t)\,\alpha(t)\,dt \tag{1.3.9}$$

subject to $Z(t)$ is bounded and measurable (1.3.10)

$$B(t)Z(t) \leq \gamma(t) + \int_0^t K(t,s)Z(s)\,ds, \quad 0 \leq t \leq T \tag{1.3.11}$$

$$Z(t) \geq 0 \qquad 0 \leq t \leq T \tag{1.3.12}$$

Dual Program (D.P)

$$\text{minimize} \quad \int_0^T \gamma(t)W(t)\,dt \tag{1.3.13}$$

subject to $W(t)$ is bounded and measurable (1.3.14)

$$B'(t)W(t) \geq \alpha(t) + \int_t^T K'(s,t)W(s)\,ds, \quad 0 \leq t \leq T \tag{1.3.15}$$

$$W(t) \geq 0 \qquad 0 \leq t \leq T \tag{1.3.16}$$

Levinson [60] has shortened the proofs by treating the continuous problem directly.

It is to be noted that a sufficient condition for duality theorem of (finite) linear programming, that both primal and dual problems be feasible is no longer sufficient to yield the duality theorem for the continuous linear programming problems. Tyndall [85] has proved that the existence of function Z and W ... feasible for their respective problems is not sufficient to guarentee the existence of optimising solutions.

In Chapter V of the present thesis we have presented the duality theorem for continuous nonlinear programming problems.

Section I deals with continuous quadratic programming problem which is :

Primal Program (P.P)

$$\text{maximize} \quad \int_0^T f(t,Z(t))dt = \int_0^T \left[a(t)Z(t)+\tfrac{1}{2}Z'(t)D(t)Z(t)\right]dt \tag{1.3.17}$$

$$\text{subject to} \quad Z(t) \text{ is bounded and measurable} \tag{1.3.18}$$

$$BZ(t) \leq c(t) + \int_0^t CZ(s)ds \qquad 0 \leq t \leq T \tag{1.3.19}$$

$$Z(t) \geq 0 \qquad 0 \leq t \leq T \tag{1.3.20}$$

Dual Program (D.P)

$$\text{minimize} \quad \int_0^T g(t,U(t),W(t))dt = \int_0^T \left[c(t)W(t)-\tfrac{1}{2}U'(t)D(t)U(t)\right]dt \tag{1.3.21}$$

$$\text{subject to} \quad W(t),\ U(t) \text{ are bounded and measurable} \tag{1.3.22}$$

$$B'W(t) \geq a(t) + D(t)U(t) + \int_t^T C'W(s)ds, \quad 0 \leq t \leq T \tag{1.3.23}$$

$$W(t) \geq 0 \qquad 0 \leq t \leq T \tag{1.3.24}$$

$D(t)$ is $N \times N$ symmetric negative semidefinite matrix for $0 \leq t \leq T$ such that for all feasible $U(t)$, $D(t)U(t)$ is bounded and measurable.

The associated linear version is

$$\text{maximize} \quad \int_0^T \left[a(t)Z(t)+Z_o'(t)D(t)Z(t)-\tfrac{1}{2}Z_o'(t)D(t)Z_o(t)\right] dt \tag{1.3.25}$$

$$\text{subject to} \quad Z(t) \text{ is bounded and measurable} \tag{1.3.26}$$

$$BZ(t) \leq c(t) + \int_0^t CZ(s)ds \quad 0 \leq t \leq T \tag{1.3.27}$$

$$Z(t) \geq 0 \qquad 0 \leq t \leq T \tag{1.3.28}$$

The constraints of primal problem and its associated linear problem are same. If Z_o is an optimal solution to the primal problem (P.P) then it is also optimal to the linear problem. Now from the duality theory of continuous linear programming given by Tyndall [85], [86] there exists optimal solution W_o to the dual of linear program, which is :

$$\text{minimize} \quad \int_0^T \left[c(t)W(t)-\tfrac{1}{2}Z_o'(t)D(t)Z_o(t)\right] dt \tag{1.3.29}$$

subject to $W(t)$ is bounded and measurable (1.3.30)

$$B'W(t) \geq a(t)+D(t)Z_o(t)+\int_t^T C'W(s)ds, \quad 0 \leq t \leq T \tag{1.3.31}$$

$$W(t) \geq 0 \qquad 0 \leq t \leq T \tag{1.3.32}$$

and

$$\int_0^T \left[C(t)W_o(t)-\tfrac{1}{2}Z_o'(t)D(t)Z_o(t)\right]dt = \int_0^T \left[a(t)Z_o(t)+\tfrac{1}{2}Z_o'(t)D(t)Z_o(t)\right] dt \tag{1.3.33}$$

Using this the duality theorem for quadratic program is proved.

In section II we have considered a continuous convex programming problem.

<u>Primal Program</u> (P.P)

$$\text{maximize} \quad \int_0^T f(t,Z(t))dt \tag{1.3.34}$$

subject to $\quad Z(t)$ is bounded and measurable (1.3.35)

$$BZ(t) \leq c(t) + \int_0^t CZ(s)ds \quad 0 \leq t \leq T \tag{1.3.36}$$

$$Z(t) \geqslant 0 \qquad 0 \leq t \leq T \tag{1.3.37}$$

Dual Program (D.P)

$$\text{minimise} \quad \int_0^T \left[f(t,U(t)) - U'(t) \nabla f(t,U(t)) + c'(t)W(t)\right] dt \tag{1.3.38}$$

subject to $\quad W(t), U(t)$ are bounded and measurable (1.3.39)

$$B'W(t) \geqslant \nabla f(t,U(t)) + \int_t^T C'W(s)ds \quad 0 \leq t \leq T \tag{1.3.40}$$

$$W(t) \geqslant 0 \qquad 0 \leq t \leq T \tag{1.3.41}$$

$f(t,Z(t))$ is differentiable concave function of $Z(t)$ for $0 \leq t \leq T$.

The proof of duality theorem for this pair of programs is also based on the duality theory of linear programming. Associated to this a linear version is taken and using the duality theory of linear programming given by Tyndall [85] [86] existence of optimal solution to the above pair of programs is shown.

CHAPTER - II

Symmetric Dual and Self Dual Quadratic Programming Problems in Complex Space

The purpose of this chapter is to present the symmetric dual and self dual quadratic programs in complex space. This chapter is divided into two sections. Section I deals with the pair of symmetric dual quadratic programs in complex space. Some duality theorems are proved for this pair of symmetric programs. Duality theorem is proved by adopting the technique of Dorn [21] and using the duality theorem for linear programming in complex space [59]. The self-dual linear programs and formal self-dual quadratic programs in complex space are defined and the self duality theorem is proved for each program seperately.

Section II is devoted to the study of self dual quadratic programs in complex space and fundamental theorem in quadratic programming in complex space. The self dual quadratic program of Dorn [24] is generalized and the fundamental theorem in quadratic programming given by Cottle [15] is extended to quadratic programming in complex space. The main aim of this section is to generalise the properties of real symmetric positive definite and positive semi-definite matrices to Hermitian matrices whose Hermitian forms are positive definite and positive semi-definite respectively, since they are found to be of great use in problems considered in the present chapter.

SECTION - I

Symmetric Dual Quadratic Program in Complex Space*

Symmetric dual complex quadratic programming problems may be stated as follows:

Primal Program (P.P.B)

minimize $\quad F(Z,W) = \text{Re}\left[\tfrac{1}{2}W^{*}DW+\tfrac{1}{2}Z^{*}CZ+p^{*}Z\right]$ (2.1.1)

subject to $\quad \left|\text{Arg}\,(DW+AZ-b)\right| \leq \beta$ (2.1.2)

$\left|\text{Arg}\,Z\right| \leq \alpha$ (2.1.3)

Dual Program $(D.P.\tilde{B})$

maximize $\quad G(\lambda,\mu) = \text{Re}\left[-\tfrac{1}{2}\mu^{*}D\mu-\tfrac{1}{2}\lambda^{*}C\lambda+b^{*}\mu\right]$ (2.1.4)

subject to $\quad \left|\text{Arg}\,(-A^{*}\mu+C\lambda+p)\right| \leq \frac{\pi}{2}-\alpha$ (2.1.5)

$\left|\text{Arg}\,\mu\right| \leq \frac{\pi}{2}-\beta$ (2.1.6)

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A, C, D, b, p, are as defined in Chapter I. If in the above programs B and $\tilde{B}$ we have all A, C, D, b, p with real entries and $\alpha = 0$, $\beta = \frac{\pi}{2}$ then these programs reduce to the symmetric dual quadratic programs considered by Cottle [14].

A feasible $(\tilde{\tilde{Z}}, \tilde{\tilde{W}})$ which solves the primal problem in such a manner that for all feasible (Z, W)

$$\operatorname{Re}(\tfrac{1}{2}\tilde{\tilde{W}}^* D\tilde{\tilde{W}} + \tfrac{1}{2}\tilde{\tilde{Z}}^* C\tilde{\tilde{Z}} + p^*\tilde{\tilde{Z}}) \leq \operatorname{Re}\left[\tfrac{1}{2}W^* DW + \tfrac{1}{2}Z^* CZ + p^* Z\right]$$

is said to be optimal feasible. Similarly a $(\tilde{\tilde{\lambda}}, \tilde{\tilde{\mu}})$ which solves the dual problem in such a manner that for all feasible (λ, μ),

$$\operatorname{Re}\left[-\tfrac{1}{2}\tilde{\tilde{\mu}}^* D\tilde{\tilde{\mu}} - \tfrac{1}{2}\tilde{\tilde{\lambda}}^* C\tilde{\tilde{\lambda}} + b^*\tilde{\tilde{\mu}}\right] \geq \operatorname{Re}\left[-\tfrac{1}{2}\mu^* D\mu - \tfrac{1}{2}\lambda^* C\lambda + b^*\mu\right]$$

is said to be optimal feasible for dual problem.

Theorem 2.1A (Weak Duality Theorem)

If (Z, W) is feasible for problem B and (λ, μ) is feasible for problem $\tilde{\tilde{B}}$ then

$$\text{minimum} \quad \operatorname{Re}\left[\tfrac{1}{2}W^* DW + \tfrac{1}{2}Z^* CZ + p^* Z\right]$$
$$\geq \text{maximum} \operatorname{Re}\left[-\tfrac{1}{2}\mu^* D\mu - \tfrac{1}{2}\lambda^* C\lambda + b^*\mu\right] \qquad (2.1.7)$$

Proof. Since C and D are positive semi-definite Hermitian matrices, therefore for all Z and λ we have

$$(Z - \lambda)^* C (Z - \lambda) \geq 0$$

$$Z^* CZ + \lambda^* C\lambda \geq 2Z^* C\lambda \qquad (2.1.8)$$

and for all W, μ

$$(W - \mu)^* D (W - \mu) \geq 0$$

$$W^* DW + \mu^* D\mu \geq 2W^* D\mu \qquad (2.1.9)$$

Let (Z, W) be feasible for primal Problem B and (λ, μ) be feasible for dual problem $\tilde{\tilde{B}}$ then from (2.1.2) and (2.1.6)

$$\operatorname{Re}\left[(DW + AZ - b), \mu\right] \geqslant 0 \qquad (2.1.10)$$

and from (2.1.3) and (2.1.5)

$$\operatorname{Re}\left[Z, (-A^*\mu + C\lambda + p)\right] \geqslant 0 \qquad (2.1.11)$$

Using $(AZ, \mu) = (Z, A^*\mu)$ we have

$$\left[Z, (-A^*\mu + C\lambda + p)\right] + \left[(DW + AZ - b), \mu\right] + b^*\mu$$

$$= Z^*C\lambda + W^*D\mu + Z^*p \qquad (2.1.12)$$

Using (2.1.10) (2.1.11) in (2.1.12) we get

$$\operatorname{Re}(b^*\mu) \leqslant \operatorname{Re}\left[Z^*C\lambda + W^*D\mu + p^*Z\right] \qquad (2.1.13)$$

$$\leqslant \operatorname{Re}\left[\tfrac{1}{2}Z^*CZ + \tfrac{1}{2}\lambda^*C\lambda + \tfrac{1}{2}W^*DW + \tfrac{1}{2}\mu^*D\mu + p^*Z\right]$$

by (2.1.8) and (2.1.9)

or $$\operatorname{Re}\left[-\tfrac{1}{2}\mu^*D\mu - \tfrac{1}{2}\lambda^*C\lambda + b^*\mu\right] \leqslant \operatorname{Re}\left[\tfrac{1}{2}Z^*CZ + \tfrac{1}{2}W^*DW + p^*Z\right] \qquad (2.1.14)$$

or $$\min \operatorname{Re}\left[\tfrac{1}{2}Z^*CZ + \tfrac{1}{2}W^*DW + p^*Z\right]$$

$$\geqslant \max \operatorname{Re}\left[-\tfrac{1}{2}\mu^*D\mu - \tfrac{1}{2}\lambda^*C\lambda + b^*\mu\right]$$

Hence the proof

<u>Theorem 2.1B (Optimality Condition)</u>

If the respective problem possess feasible $(\tilde{Z}, \tilde{W})$ and $(\tilde{\lambda}, \tilde{\mu})$ for which

$$\operatorname{Re}\left[-\tfrac{1}{2}\tilde{\mu}^*D\tilde{\mu} - \tfrac{1}{2}\tilde{\lambda}^*C\tilde{\lambda} + b^*\tilde{\mu}\right] = \operatorname{Re}\ \tfrac{1}{2}\left[\tilde{Z}^*C\tilde{Z} + \tfrac{1}{2}\tilde{W}^*D\tilde{W} + p^*\tilde{Z}\right] (2.1.15)$$

then $(\tilde{Z}, \tilde{W})$ and $(\tilde{\lambda}, \tilde{\mu})$ are optimal feasible.

<u>Proof</u>. By (2.1.14) for any feasible (Z, W) and $(\tilde{\lambda}, \tilde{\mu})$

$$\operatorname{Re}\left[\tfrac{1}{2}Z^*CZ + \tfrac{1}{2}W^*DW + p^*Z\right] \geqslant \operatorname{Re}\left[-\tfrac{1}{2}\tilde{\mu}^*D\tilde{\mu} - \tfrac{1}{2}\tilde{\lambda}^*C\tilde{\lambda} + b^*\tilde{\mu}\right] \qquad (2.1.16)$$

using (2.1.15) in (2.1.16)

$$\operatorname{Re}\left[\tfrac{1}{2}Z^*CZ + \tfrac{1}{2}W^*DW + p^*Z\right] \geq \operatorname{Re}\left[-\tfrac{1}{2}\tilde{\mu}^*D\tilde{\mu} - \tfrac{1}{2}\tilde{\lambda}^*C\tilde{\lambda} + b^*\tilde{\mu}\right] =$$

$$= \operatorname{Re}\left[\tfrac{1}{2}\tilde{Z}^*C\tilde{Z} + \tfrac{1}{2}\tilde{W}^*D\tilde{W} + p^*\tilde{Z}\right]$$

or $$\operatorname{Re}\left[\tfrac{1}{2}Z^*CZ + \tfrac{1}{2}W^*DW + p^*Z\right] \geq \operatorname{Re}\left[\tfrac{1}{2}\tilde{Z}^*C\tilde{Z} + \tfrac{1}{2}\tilde{W}^*D\tilde{W} + p^*\tilde{Z}\right]$$

Therefore $(\tilde{Z}, \tilde{W})$ is optimal for primal problem B similarly it can be shown that $(\tilde{\lambda}, \tilde{\mu})$ is optimal for dual problem $\tilde{\tilde{B}}$.

<u>Lemma</u> I. If D is positive semidefinite Hermitian matrix then for any complex vectors Z and W

$$\operatorname{Re}\left[W^*DW - Z^*DZ\right] \geq \operatorname{Re}\left[2(W-Z)^* DZ\right]$$

<u>Proof</u>. Since D is positive semi-definite Hermitian matrix we have

$$(W - Z)^* D (W - Z) \geq 0$$

$$W^*DW + Z^*DZ \geq Z^*DW + W^*DZ$$

Substracting $2Z^*DZ$ from both sides

$$W^*DW - Z^*DZ \geq Z^*DW + W^*DZ - 2Z^*DZ$$

$$W^*DW - Z^*DZ \geq Z^*DW - W^*DZ + 2(W - Z)^* DZ \tag{2.1.17}$$

Since the first two terms on the right hand side of (2.1.17) are conjugate and the difference of two conjugate numbers is pure imaginary therefore one gets

$$\operatorname{Re}(W^*DW - Z^*DZ) \geq \operatorname{Re}\left[2(W - Z)^* DZ\right]$$

<u>Lemma</u> 2. If Z_o, W_o; Z_1, W_1 are feasible solutions of the primal problem B then

$$Z_2 = Z_1 + \theta (Z_o - Z_1)$$

$$W_2 = W_1 + \theta (W_o - W_1)$$

for $0 < \theta < 1$ are also feasible.

Proof. $\left|\text{Arg } Z_o\right| \leq \alpha$ implies $\left|\text{Arg } \theta Z_o\right| \leq \alpha$

$\left|\text{Arg } Z_1\right| \leq \alpha$ implies $\left|\text{Arg } (1 - \theta) Z_1\right| \leq \alpha$

Now by lemma 2 of [42]

$$\left|\text{Arg } \left[\theta Z_o + (1 - \theta) Z_1\right]\right| = \left|\text{Arg } Z_2\right| \leq \alpha$$

Similarly

$\left|\text{Arg } (DW_o + AZ_o - b)\right| \leq \beta$

implies $\left|\text{Arg } (D \theta W_o + A \theta Z_o - b \theta)\right| \leq \beta$

$\left|\text{Arg } (DW_1 + AZ_1 - b)\right| \leq \beta$

implies $\left|\text{Arg } \left[D(1 - \theta) W_1 + A (1 - \theta) Z_1 - b(1 - \theta)\right]\right| \leq \beta$

and from lemma 2 of [42]

$$\left|\text{Arg } (D\left[\theta W_o + (1 - \theta) W_1\right] + A\left[\theta Z_o + (1 - \theta) Z_1\right] - b)\right|$$

$$= \left|\text{Arg } (DW_2 + AZ_2 - b)\right| \leq \beta$$

Hence the proof.

Lemma 3. If (Z_o, W_o) is an optimal solution of primal problem B then it is an optimal solution of linear program (L.P.)

(LP) minimize $f(Z, W) = \text{Re}\left[W_o^* DW + Z_o^* CZ + p^* Z\right]$

Subject to $\left|\text{Arg } (DW + AZ - b)\right| \leq \beta$

$\left|\text{Arg } Z\right| \leq \alpha$

<u>Proof</u>. The constraint set of primal problem B and (LP) are identical. Suppose there exists (Z_1,W_1) such that

$$\mathrm{Re}\left[W_o^* DW_1 + Z_o^* CZ_1 + p^* Z_1 - W_o^* DW_o - Z_o^* CZ_o - p^* Z_o\right] < 0$$

or $$\mathrm{Re}\left[W_o^* D\,(W_1 - W_o) + Z_o^* C\,(Z_1 - Z_o) + p^*(Z_1 - Z_o)\right] < 0 \tag{2.1.18}$$

Define $Z_2 = (1-\theta)Z_o + \theta Z_1 = Z_o + \theta\,(Z_1 - Z_o)$

$W_2 = (1-\theta)W_o + \theta W_1 = W_o + \theta\,(W_1 - W_o)$

for $0 < \theta < 1$

From lemma 2 (Z_2,W_2) is also feasible for primal problem B and (L.P.)

Now consider

$$\mathrm{Re}\ \left(\tfrac{1}{2}W_2^* DW_2 + \tfrac{1}{2}Z_2^* CZ_2 + p^* Z_2\right) - \left(\tfrac{1}{2}\,W_o^* DW_o + \tfrac{1}{2}Z_o^* CZ_o + p^* Z_o\right)$$

$$= \mathrm{Re}\Big\{\tfrac{1}{2}\left[W_o + \theta\,(W_1 - W_o)\right]^* D(W_o + \theta\,(W_1 - W_o)) + \tfrac{1}{2}\Big[(Z_o + \theta(Z_1 - Z_o))^*$$

$$C\,(Z_o + \theta\,(Z_1 - Z_o))\Big]$$

$$+ p^*\left(Z_o + \theta\,(Z_1 - Z_o)\right) - \left[\tfrac{1}{2}\,(W_o^* DW_o) + \tfrac{1}{2}Z_o^* CZ_o + p^* Z_o\right]\Big\}$$

$$= \mathrm{Re}\Big\{\tfrac{1}{2}W_o^* DW_o + \frac{\theta^2}{2}\,(W_1 - W_o)^* D(W_1 - W_o) + \theta\,(W_1 - W_o)^* DW_o + \tfrac{1}{2}Z_o^* CZ_o$$

$$+ \frac{\theta^2}{2}(Z_1 - Z_o)^*\,C(Z_1 - Z_o) + \theta\,(Z_1 - Z_o)^* CZ_o + p^* Z_o + \theta\,p^*(Z_1 - Z_o)$$

$$- \tfrac{1}{2}\,W_o^* DW_o - \tfrac{1}{2}Z_o^* CZ_o - \tfrac{1}{2}\,p^* Z_o\Big\}$$

$$= \theta\Big\{\mathrm{Re}\,\left[W_o^*\,D(W_1 - W_o) + (Z_o^* C + p^*)\,(Z_1 - Z_o)\right]$$

$$+ \frac{\theta}{2}\left[(W_1 - W_o)^*\,D(W_1 - W_o) + (Z_1 - Z_o)^*\,C(Z_1 - Z_o)\right]\Big\} \tag{2.1.19}$$

By using (2.1.18) and choosing θ sufficiently small positive the right hand side of (2.1.19) can be made negative. This contradicts the assumption of optimality of (Z_o, W_o) in the primal problem B. Therefore (Z_o, W_o) must be optimal for problem (L.P.)

Theorem (2.1C) (Duality Theorem)

If (Z_o, W_o) solves the primal problem B then there exists a vector μ_o such that (Z_o, μ_o) solves the dual problem $\tilde{\tilde{B}}$, with $\mu_o^* D = W_o^* D$ and $F(Z_o, W_o) = G(Z_o, \mu_o)$. Conversely, if (λ_o, μ_o) solves the dual problem $\tilde{\tilde{B}}$ then there exists a vector Z_o such that (Z_o, μ_o) solves (PP) with $\lambda_o^* C = Z_o^* C$ and $G(\lambda_o, \mu_o) = F(Z_o, \mu_o)$. Moreover the pairs (Z_o, μ_o) and (λ_o, W_o) solve both the problems.

Proof. By symmetry of the problems the converse need not be proved. By lemma 3 (Z_o, W_o) is optimal for (LP). Then by duality theorem of linear programming in complex space given by Levinson [59] there exists a vector μ_o such that

$$\left|\operatorname{Arg}\left(-A^* \mu_o + CZ_o + p\right)\right| \leq \frac{\pi}{2} - \alpha \tag{2.1.20}$$

$$\left|\operatorname{Arg} \mu_o\right| \leq \frac{\pi}{2} - \beta \tag{2.1.21}$$

$$\mu_o^* D = W_o^* D \tag{2.1.22}$$

and $$\operatorname{Re}\left(b^* \mu_o\right) = \operatorname{Re}\left[W_o^* D W_o + Z_o^* C Z_o + p^* Z_o\right] \tag{2.1.23}$$

From (2.1.20) and (2.1.21) it is clear that (Z_o, μ_o) is feasible for dual problem $\tilde{\tilde{B}}$.

Since D is Hermitian we have from (2.1.22)

$$\mu_o^* D \mu_o = W_o^* D \mu_o = \mu_o^* D W_o = W_o^* D W_o \tag{2.1.24}$$

Using (2.1.23) and (2.1.24)

$$F(Z_o,W_o) = \text{Re}\left[\tfrac{1}{2}W_o^*DW_o + \tfrac{1}{2}Z_o^*CZ_o + p^*Z_o\right]$$

$$= \text{Re}\left[-\tfrac{1}{2}W_o^*DW_o - \tfrac{1}{2}Z_o^*CZ_o + b^*\mu_o\right]$$

$$= \text{Re}\left[-\tfrac{1}{2}\mu_o^*D\mu_o - \tfrac{1}{2}Z_o^*CZ_o + b^*\mu_o\right]$$

$$= G(Z_o, \mu_o)$$

$$F(Z_o,W_o) = G(Z_o, \mu_o) \qquad (2.1.25)$$

Now from theorem (2.1B) we know that if (Z_o,W_o) solves primal problem B and if there exists (λ_o, μ_o) feasible for dual problem $\tilde{\tilde{B}}$ such that $F(Z_o,W_o) = G(\lambda_o, \mu_o)$ then (λ_o, μ_o) is optimal for $\tilde{\tilde{B}}$ therefore (Z_o, μ_o) is optimal feasible for dual problem $\tilde{\tilde{B}}$, therefore it solves $\tilde{\tilde{B}}$.

From (2.1.25) and (2.1.22) we have

$$F(Z_o,W_o) = G(Z_o, \mu_o)$$

$$\mu_o^*D = W_o^*D$$

Therefore (Z_o, μ_o) is feasible for primal problem B and

$$G(Z_o, \mu_o) = \text{Re}\left[-\tfrac{1}{2}\mu_o^*D\mu_o - \tfrac{1}{2}Z_o^*CZ_o + b^*\mu_o\right]$$

using (2.1.23)

$$= \text{Re}\left[-\tfrac{1}{2}\mu_o^*D\mu_o + W_o^*DW_o + \tfrac{1}{2}Z_o^*CZ_o + p^*Z_o\right]$$

using (2.1.24)

$$= \text{Re}\left[\tfrac{1}{2}\mu_o^*D\mu_o + \tfrac{1}{2}Z_o^*CZ_o + p^*Z_o\right]$$

$$= F(Z_o, \mu_o)$$

i.e. $$G(Z_o, \mu_o) = F(Z_o, \mu_o) \qquad (2.1.26)$$

(2.1.26) implies that (Z_o, μ_o) is optimal for primal problem therefore, (Z_o, μ_o) solves B.

Hence the proof.

<u>Theorem 2.1D</u> (<u>Complementary Slackness Theorem</u>).

If (Z_o, μ_o) is a joint solution of B and $\tilde{\tilde{B}}$ then

$$\mathrm{Re}\left[Z_o, (-A^* \mu_o + CZ_o + p)\right] = 0$$

$$\mathrm{Re}\left[(D\mu_o + AZ_o - b), \mu_o\right] = 0$$

<u>Proof</u>. Since (Z_o, μ_o) is joint solution therefore,

$$F(Z_o, \mu_o) = G(Z_o, \mu_o)$$

that is $\mathrm{Re}\left[\frac{1}{2}\mu_o^* D\mu_o + \frac{1}{2}Z_o^* CZ_o + p^* Z_o\right] = \mathrm{Re}\left[-\frac{1}{2}\mu_o^* D\mu_o - \frac{1}{2}Z_o^* CZ_o + b^*\mu_o\right]$

$$(2.1.27)$$

From $(AZ_o, \mu_o) = (Z_o, A^*\mu_o)$ it follows that

$$\left[Z_o, (-A^*\mu_o + CZ_o + p)\right] + \left[(D\mu_o + AZ_o - b), \mu_o\right] + b^*\mu_o \qquad (2.1.28)$$

$$= \left[Z_o^* CZ_o + \mu_o^* D\mu_o + Z_o^* p\right]$$

From (2.1.27) $\mathrm{Re}\,(b^*\mu_o) = \mathrm{Re}\left[\mu_o^* D\mu_o + Z_o^* CZ_o + p^* Z_o\right]$

using (2.1.27) in (2.1.28) we get

$$\mathrm{Re}\left[Z_o, (-A^*\mu_o + CZ_o + p)\right] + \mathrm{Re}\left[(D\mu_o + AZ_o - b), \mu_o\right] = 0 \qquad (2.1.29)$$

But from the feasibility of (Z_o, μ_o) for primal B and dual problem $\tilde{\tilde{B}}$ we have

$$\mathrm{Re}\left[Z_o, (-A^*\mu_o + CZ_o + p)\right] \geqslant 0 \qquad (2.1.30)$$

and

$$\mathrm{Re}\left[(D\mu_o + AZ_o - b), \mu_o\right] \geqslant 0 \qquad (2.1.31)$$

(2.1.30) and (2.1.31) with (2.1.29) give

$$\mathrm{Re}\left[Z_o, (-A^*\mu_o + CZ_o + p)\right] = 0$$

$$\mathrm{Re}\left[(D\mu_o + AZ_o - b), \mu_o\right] = 0$$

Hence the proof.

(2.1.a) Self Dual Linear Program in Complex Space

Consider the following linear programming problem in complex space

Primal Problem (P.P.(B_1)) Find a Z if there is any such that

$$\left|\mathrm{Arg}\ (AZ + b)\right| \leq \frac{\pi}{2} - \alpha \tag{2.1.32}$$

$$\left|\mathrm{Arg}\ Z\right| \leq \alpha \tag{2.1.33}$$

and $\mathrm{Re}\ (b^*Z)$ is minimized (2.1.34)

The dual of this is the following

Dual Program D.P. $(\tilde{\bar{B}}_1)$ Find a W if there is any such that

$$\left|\mathrm{Arg}\ (-A^*W + b)\right| \leq \frac{\pi}{2} - \alpha \tag{2.1.35}$$

$$\left|\mathrm{Arg}\ W\right| \leq \alpha \tag{2.1.36}$$

and $\mathrm{Re}\ (-b^*W)$ is maximized (2.1.37)

Theorem (2.1E) (Self Duality Theorem)

If A is skew Hermitian, then, (B_1) and $(\tilde{\bar{B}}_1)$ are identical, and if (B_1) and $(\tilde{\bar{B}}_1)$ are dual programs and Z_o is optimal solution of (B_1) and W_o is optimal solution of $(\tilde{\bar{B}}_1)$, then $\mathrm{Re}\ (b^*Z_o) = \mathrm{Re}\ (-b^*W_o) = 0$.

Proof. Since $A = -A^*$ (A is skew Hermitian), the program $(\tilde{\bar{B}}_1)$ can be written as follows:

$$\left|\text{Arg }(AW + b)\right| \leq \frac{\pi}{2} - \alpha$$

$$\left|\text{Arg } W\right| \leq \alpha$$

$\text{Re }(b^*W)$ is minimized.

This is just the program (B_1).

From (2.1.32) and (2.1.33) we have for feasible Z_o

$$\text{Re}\left[Z_o, (AZ_o + b)\right] \geqslant 0$$

$$\text{Re}\left[Z_o^*AZ_o + b^*Z_o\right] \geqslant 0$$

But $Z_o^*AZ_o$ is skew Hermitian form, therefore purely imaginary or zero

$$\text{Re}\left[b^*Z_o\right] \geqslant 0 \tag{2.1.38}$$

Similarly, from (2.1.35) and (2.1.36), for feasible W_o,

$$\text{Re}\left[(-A^*W_o + b), W_o\right] \geqslant 0$$

or $\text{Re }(b^*W_o) \geqslant 0$

$$\text{Re }(-b^*W_o) \leqslant 0 \tag{2.1.39}$$

If Z_o and W_o are optimal solutions then from duality theorem given by Levinson [59]

$$\text{Re }(-b^*W_o) = \text{Re }(b^*Z_o)$$

using (2.1.38) and (2.1.39)

$$\text{Re }(-b^*W_o) = \text{Re }(b^*Z_o) = 0.$$

Hence the proof.

(2.1b) <u>Formal Self Dual Quadratic Program in Complex Space</u>.

Consider the following quadratic programs in complex space.

<u>Primal Program P.P.</u>(B_2) minimize $\phi(Z,W) = \text{Re}\left[\frac{1}{2}W^*CW + \frac{1}{2}Z^*CZ + p^*Z\right]$ (2.1.40)

subject to $\left|\operatorname{Arg}(CW + AZ + p)\right| \leq \frac{\pi}{2} - \alpha$ (2.1.41)

$\left|\operatorname{Arg} Z\right| \leq \alpha$ (2.1.42)

<u>Dual Program D.P.</u> $(\tilde{\tilde{B}}_2)$

maximize $\psi(\lambda,\mu) = \operatorname{Re}\left[-\tfrac{1}{2}\mu^{*}C\mu - \tfrac{1}{2}\lambda^{*}C\lambda - p^{*}\mu\right]$ (2.1.43)

subject to $\left|\operatorname{Arg}(-A^{*}\mu + C\lambda + p)\right| \leq \frac{\pi}{2} - \alpha$ (2.1.44)

$\left|\operatorname{Arg}\mu\right| \leq \alpha$ (2.1.45)

<u>Theorem 2.1F</u>. If A is skew Hermitian and C is Hermitian positive semi definite, then the program (B_2) is self dual. Moreover if (B_2) has an optimal feasible solution then min $\operatorname{Re}\left[\tfrac{1}{2}W^{*}CW + \tfrac{1}{2}Z^{*}CZ + p^{*}Z\right] = 0$.

<u>Proof</u>. Since A is skew Hermitian the program $(\tilde{\tilde{B}}_2)$ can also be written as

minimize $-\psi(\lambda,\mu) = \operatorname{Re}\left[\tfrac{1}{2}\mu^{*}C\mu + \tfrac{1}{2}\lambda^{*}C\lambda + p^{*}\mu\right]$

subject to $\left|\operatorname{Arg}(A\mu + C\lambda + p)\right| \leq \frac{\pi}{2} - \alpha$

$\left|\operatorname{Arg}\mu\right| \leq \alpha$

This is identical to (B_2).

Since $Z^{*}AZ$ is skew Hermitian form, therefore, it is purely imaginary or zero.

Let (Z,W) be feasible for B_2 then from (2.1.41) and (2.1.42)

$$\operatorname{Re}\left[Z, (CW + AZ + p)\right] \geq 0$$

$$\operatorname{Re}(Z^{*}CW + Z^{*}AZ + Z^{*}p) \geq 0$$

$$\operatorname{Re}(Z^{*}CW + p^{*}Z) \geq 0 \quad \text{or} \quad \operatorname{Re}(W^{*}CZ + p^{*}Z) \geq 0 \tag{2.1.46}$$

From positive semi-definiteness of C we obtain

$$\operatorname{Re}\left(\tfrac{1}{2}W^{*}CW + \tfrac{1}{2}Z^{*}CZ\right) \geq \operatorname{Re}(W^{*}CZ) \tag{2.1.47}$$

Now $\phi(Z,W) = \text{Re}\left[\frac{1}{2}W^*CW + \frac{1}{2}Z^*CZ + p^*Z\right]$

$$\geqslant \text{Re}\,(W^*CZ + p^*Z) \quad \text{by} \quad (2.1.47)$$

$$\phi(Z,W) \geqslant 0 \qquad \text{by } (2.1.46)$$

The program (B_2) is solvable since, $\phi(Z,W)$ is bounded below.

Therefore the dual program $(\tilde{\tilde{B}}_2)$ is also solvable and for all feasible (λ, μ) we have

$$\psi(\lambda,\mu) = \text{Re}\left[-\frac{1}{2}\mu^*C\mu - \frac{1}{2}\lambda^*C\lambda - p^*\mu\right] \leqslant 0$$

Then

$$0 \leqslant \text{Re}\left[\frac{1}{2}W^*CW + \frac{1}{2}Z^*CZ + p^*Z\right] = \text{Re}\left[-\frac{1}{2}\mu^*C\mu - \frac{1}{2}\lambda^*C\lambda - p^*\mu\right] \leqslant 0$$

Therefore, by duality theorem for optimal solution

i.e. $\text{Re}\left[\frac{1}{2}W^*CW + \frac{1}{2}Z^*CZ + p^*Z\right] = \text{Re}\left[-\frac{1}{2}\mu^*C\mu - \frac{1}{2}\lambda^*C\lambda - p^*\mu\right] = 0.$

Hence, (B_2) is formally self dual.

Hence the proof.

<u>(2.10) Composite Quadratic Program in Complex Space.</u>

Consider the composite program of (B) and $(\tilde{\tilde{B}})$

minimize $\eta = \text{Re}\left[\frac{1}{2}W^*DW + \frac{1}{2}Z^*CZ + \frac{1}{2}\mu^*D\mu + \frac{1}{2}\lambda^*C\lambda + p^*Z - b^*\mu\right]$

subject to
$$|\text{Arg}\,(DW + AZ - b)| \leq \beta$$
$$|\text{Arg}\,(-A^*\mu + C\lambda + p)| \leq \frac{\pi}{2} - \alpha$$
$$|\text{Arg}\,Z| \leq \alpha$$
$$|\text{Arg}\,\mu| \leq \frac{\pi}{2} - \beta$$

Taking $A_1 = \begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix}$, $C_1 = \begin{pmatrix} D & 0 \\ 0 & C \end{pmatrix}$

$$q = \begin{pmatrix} -b \\ p \end{pmatrix}, \quad X = \begin{pmatrix} \mu \\ z \end{pmatrix}, \quad Y = \begin{pmatrix} w \\ \lambda \end{pmatrix}, \quad \gamma = \begin{vmatrix} \frac{\pi}{2}-\beta \\ \alpha \end{vmatrix}$$

minimize $\quad \operatorname{Re}\left[\frac{1}{2} Y^*C_1Y + \frac{1}{2}X^*C_1X + q^*X\right]$

subject to $\left|\operatorname{Arg}(C_1Y + A_1X + q)\right| \leq \frac{\pi}{2} - \gamma$

$$\left|\operatorname{Arg} X\right| \leq \gamma$$

This program is of the form B_2 which is self dual.

SECTION - II

Self Dual Quadratic Program in Complex Space

The self dual quadratic programming problem in complex space is defined as follows

Primal Program P.P. (B)

minimize $\quad f(Z) = \operatorname{Re}\left[Z^*CZ + p^*Z\right] \qquad (2.2.1)$

subject to $\left|\operatorname{Arg}(CZ + p)\right| \leq \frac{\pi}{2} - \alpha \qquad (2.2.2)$

$\left|\operatorname{Arg} Z\right| \leq \alpha \qquad (2.2.3)$

Dual Program D.P. ($\tilde{\bar{B}}$)

maximize $\quad g(\lambda, \mu) = \operatorname{Re}\left[-p^*\mu - \lambda^* C\lambda\right] \qquad (2.2.4)$

subject to $\left|\operatorname{Arg}(-C^*\mu + 2C\lambda + p)\right| \leq \frac{\pi}{2} - \alpha \qquad (2.2.5)$

$\left|\operatorname{Arg} \mu\right| \leq \alpha \qquad (2.2.6)$

If C, p have real entries and $\alpha = 0$ the problem B and $\tilde{\bar{B}}$ reduce to the self dual convex quadratic program in real space given by Dorn [24] .

Theorem (2.2A). If there is a Z satisfying (2.2.2) and (2.2.3) then the program B is self dual.

Proof. Let there is a Z satisfying (2.2.2) and (2.2.3) then

$\text{Re}\ (Z^*CZ + p^*Z) \geqslant 0$

Since minimum $f(Z) = \text{Re}\ (Z^*CZ + p^*Z) \geqslant 0$ is bounded below therefore B has an optimal solution.

Now by the duality theorem given by Hanson and Mond [42] if B has an optimal solution then $\tilde{\tilde{B}}$ also has an optimal solution and moreover

$$\text{Re}\ (Z^*CZ + p^*Z) = \text{Re}\ (-p^*\mu - \lambda^* C \lambda)$$

using (2.2.5) and (2.2.6)

$$\text{Re}\ \left[-\mu^* C \mu + 2\mu^* C \lambda + p^*\mu\right] \geqslant 0$$

$$\text{Re}\ (-p^*\mu) \leqslant \text{Re}\ \left[-\mu^* C\mu + 2\mu^* C\lambda\right] \tag{2.2.7}$$

$$g(\lambda, \mu) = \text{Re}\ \left[-p^*\mu - \lambda^* C \lambda\right]$$

using (2.2.7)

$$\leqslant \text{Re}\ \left[-\mu^* C\mu + 2\mu^* C\lambda - \lambda^* C\lambda\right]$$

$$= -\text{Re}\ \left[\mu^* C \mu - 2\mu^* C\lambda + \lambda^* C\lambda\right]$$

$$= -\text{Re}\ \left[(\mu-\lambda)^* C (\mu-\lambda)\right] \leq 0$$

(Since C is positive definite Hermitian matrix)

$$g(\lambda, \mu) = \text{Re}\ \left[-p^*\mu - \lambda^* C \lambda\right] \leq 0$$

$$0 \leqslant \text{minimum}\ f(Z) = \text{Re}\ (Z^*CZ + p^*Z)$$

$$= \text{Re}\ \left[-p^*\mu - \lambda^* C\lambda\right]$$

$$= \text{maximum}\ g(\lambda, \mu) \leq 0$$

As C is positive definite Hermitian matrix

maximum $g(\lambda, \mu) = \text{Re}\ \left[-p^*\mu - \lambda^* C\lambda\right] = 0$ if and only if $\lambda = \mu$.

The constraints $\lambda = \mu$ are added to (2.2.5) and (2.2.6) and the equivalent problem is obtained. The equivalent problem so obtained is

$$\text{maximize} \quad g(\lambda, \mu) = \operatorname{Re}\left[-p^{*}\mu - \lambda^{*} C \lambda\right]$$

$$\text{subject to} \quad \left|\operatorname{Arg}\left(-C^{*}\mu + 2C\lambda + p\right)\right| \leq \frac{\pi}{2} - \alpha$$

$$\left|\operatorname{Arg} \mu\right| \leq \alpha$$

$$\lambda = \mu$$

Now eliminating λ we get

$$\text{maximize} \quad g(\mu) = \operatorname{Re}\left[-p^{*}\mu - \mu^{*} C \mu\right] = \text{Maximum}\left[-f(\mu)\right]$$

$$= -\text{minimum } f(\mu)$$

$$\text{subject to} \quad \left|\operatorname{Arg}\left(C\mu + p\right)\right| \leq \frac{\pi}{2} - \alpha$$

$$\left|\operatorname{Arg} \mu\right| \leq \alpha$$

which is the original program posed by (2.2.1) (2.2.2) and (2.2.3).

(2.2a) <u>Fundamental Theorem for Quadratic Programming in Complex Space</u>.

Consider the following quadratic program in complex space

<u>Primal Program P.P.</u>(B_1)

$$\text{minimize} \quad \phi(Z) = \operatorname{Re}\left[Z^{*}MZ + q^{*}Z\right] \tag{2.2.8}$$

$$\text{subject to} \quad \left|\operatorname{Arg}\left(MZ + q\right)\right| \leq \frac{\pi}{2} - \gamma \tag{2.2.9}$$

$$\left|\operatorname{Arg} Z\right| \leq \gamma \tag{2.2.10}$$

If M is an nxn positive semi-definite matrix, q is a real vector, $\gamma = 0$ then the above program reduces to the program considered by Cottle [15] .

<u>Theorem (2.2B)</u> If M is positive semi-definite Hermitian matrix and there exists a $Z = Z_0$ satisfying (2.2.9), (2.2.10) and minimizing

$\mathrm{Re}\ (Z^*MZ + q^*Z)$ then minimum $\mathrm{Re}\ (Z^*MZ + q^*Z) = 0.$

<u>Proof</u>. The dual of the above program B_1 is

$$\text{D. P. } (\tilde{\tilde{B}}_1) \quad \text{maximize} \quad \psi(\lambda,\mu) = \mathrm{Re}\left[-\lambda^* M\lambda - q^*\mu\right] \tag{2.2.11}$$

$$\text{subject to} \quad \left|\mathrm{Arg}\ (-M\mu + (M+M^*)\lambda + q)\right| \leq \frac{\pi}{2} - \gamma \tag{2.2.12}$$

$$\left|\mathrm{Arg}\ \mu\right| \leq \gamma \tag{2.2.13}$$

From the duality theorem of quadratic programming in complex space given by Hanson and Mond [42] if primal problem has a solution then the dual problem also has a solution and the extreme values of the two objective functions are equal. Therefore the program $\tilde{\tilde{B}}_1$ also has a solution (λ, μ), $\lambda = Z_o$ such that

$$\text{minimum} \quad \phi(Z_o) = \text{maximum}\ \psi(\lambda,\mu)$$

From (2.2.9) and (2.2.10)

$$\mathrm{Re}\ (Z^*MZ + q^*Z) \geqslant 0$$

From (2.2.12) and (2.2.13)

$$\mathrm{Re}\left[-\mu^* M^*\mu + \mu^*(M + M^*)\lambda + q^*\mu\right] \geqslant 0$$

$$\text{or} \quad \mathrm{Re}\ (-q^*\mu) \leqslant \mathrm{Re}\ (-\mu^* M^*\mu + \mu^*(M + M^*)\lambda)$$

$$\text{Now} \quad \mathrm{Re}\ (-q^*\mu - \lambda^* M\lambda) \leq \mathrm{Re}\left[-\mu^* M\mu + \mu^*(M + M^*)\lambda - \lambda^* M\lambda\right]$$

$$= \mathrm{Re}\left[-(\lambda - \mu)^* M(\lambda - \mu)\right] \leqslant 0$$

$\mathrm{Re}\ (-q^*\mu - \lambda^* M\lambda) \leq 0$ (Since M is positive semi-definite Hermitian matrix).

$$0 \leqslant \text{minimum Re}\left[Z_o^* MZ_o + q^*Z_o\right]$$

$$= \text{maximum Re}\left[-\lambda^* M\lambda - q^*\mu\right] \leq 0$$

Therefore Minimum $\mathrm{Re}\left[Z_o^* MZ_o + q^*Z_o\right] = 0$

Hence the proof.

The above program is important because this generalize the composite program in complex space formed from a pair of symmetric dual quadratic programs in complex space.

It has been shown in [35] that the programs

P.P.(B_2) minimize $\quad \mathrm{Re}\left[\frac{1}{2}W^*DW + \frac{1}{2}Z^*CZ + p^*Z\right]$

subject to $\quad \left|\mathrm{Arg}\,(DW + AZ - b)\right| \leq \beta$

$$\left|\mathrm{Arg}\, Z\right| \leq \alpha$$

<u>Dual Program</u> D.P. $(\tilde{\tilde{B}}_2)$

maximize $\quad \mathrm{Re}\left[-\frac{1}{2}\mu^*D\mu - \frac{1}{2}\lambda^*C\lambda + b^*\mu\right]$

subject to $\quad \left|\mathrm{Arg}\,(-A^*\mu + C\lambda + p)\right| \leq \frac{\pi}{2} - \alpha$

$$\left|\mathrm{Arg}\,\mu\right| \leq \frac{\pi}{2} - \beta$$

are a dual pair and if either has an optimal solution there exists a pair of vectors which solves both programs. That is if (Z_o, W_o) is an optimal solution of B_2 there exists a vector μ_0 such that (Z_o, μ_0) solves B_2 and $\tilde{\tilde{B}}_2$. A similar property holds for optimal solutions of $\tilde{\tilde{B}}_2$. At optimal solutions the objective functions of B_2 and $\tilde{\tilde{B}}_2$ take the same value. Due to this fact we seek a solution of the composite program in complex space.

B_3 minimize $\quad \mathrm{Re}\left[W^*DW + Z^*CZ + p^*Z - b^*W\right]$

subject to $\quad \left|\mathrm{Arg}\,(DW + AZ - b)\right| \leq \beta$

$$\left|\mathrm{Arg}\,(-A^*W + CZ + p)\right| \leq \frac{\pi}{2} - \alpha$$

$$\left|\mathrm{Arg}\, Z\right| \leq \alpha$$

$$\left|\mathrm{Arg}\, W\right| \leq \frac{\pi}{2} - \beta$$

If D, C, A, p, b have real entries and $\alpha = 0$, $\beta = \frac{\pi}{2}$ the program B_3 reduces to the program considered by Cottle [15] .

Here if we take

$$M = \begin{pmatrix} C & -A^* \\ A & D \end{pmatrix} , \quad q = \begin{pmatrix} p \\ -b \end{pmatrix} , \quad X = \begin{pmatrix} Z \\ W \end{pmatrix} , \quad \gamma = \begin{pmatrix} \alpha \\ \frac{\pi}{2} - \beta \end{pmatrix}$$

then program B_3 has the form B_1, and M is positive semi-definite Hermitian matrix if and only if $A = 0$. Now if B_2 and $\tilde{B}_2$ have optimal solution the so does B_3 and any optimal solution of B_3 is an optimal solution of B_2 and $\tilde{B}_2$.

CHAPTER - III

Duality Theorem for Convex Program in Complex Space *

The purpose of this chapter is to present the duality theorem for convex program in complex space. Duality theorem is proved by adopting the technique of Dorn [25], and using the duality theorem for linear programming in complex space given by Levinson [59]. Weak Duality Theorem, Complementary Slackness Theorem, Optimality Condition are also proved for this program.

Duality theorem is proved in two parts

I Direct Duality Theorem,

II Converse Duality Theorem.

Direct duality theorem states that if primal problem is solvable then dual problem is also solvable, and the extreme values of two

objective functions are equal. Converse duality theorem states that if dual problem is solvable then the primal problem is also solvable and the extreme values of two objective functions are equal.

Direct duality theorem is proved under the assumption that Re $f(Z) = U(X,Y)$ is convex function of X and Y such that $X + iY = Z \in C_I$ where C_I is the region defined by primal constraints. The converse duality theorem requires in addition that the inverse gradient of the function $f(Z)$ has a derivative, i.e. the matrix M consisting of the partial derivatives of Φ and Ψ with respect to ξ and η exists, where

$$\zeta = f'(\mu)$$

or

$$\mu = (f')^{-1}(\zeta) = \phi(\zeta) + i\,\psi(\zeta)$$

$$\zeta = \xi + i\eta$$

<u>Duality Theory for Linear Programming in Complex Space</u>.

The complex linear programming problem is

P.P. (L_1) minimize $\operatorname{Re}(C^*Z)$

subject to $|\operatorname{Arg}(AZ-b)| \leq \beta$

$|\operatorname{Arg} Z| \leq \alpha$

The dual problem to this is

D.P. (L_1) maximize $\operatorname{Re}(b^*W)$

subject to $|\operatorname{Arg}(-A^*W+C)| \leq \frac{\pi}{2} - \alpha$

$|\operatorname{Arg}(W)| \leq \frac{\pi}{2} - \beta$

where A, b, c, α, β are defined in Chapter I. Now we shall show how the changes in primal constraints bring corresponding changes in dual constraints.

Let the vector Z be partitioned as (Z_1, Z_2) and consider the problem

P.P. (L) minimise $\operatorname{Re}\,(C_1^* Z_1 + C_2^* Z_2)$

$$\text{subject to } \left|\operatorname{Arg}(A_1 Z_2 + A_2 Z_2 - b)\right| \leq \beta$$

$$\left|\operatorname{Arg} Z_1\right| \leq \alpha$$

where C and A are similarly partitioned.

It is easy to see that for any vector Z_2 there exists vectors S, Q and γ such that

$$Z_2 = S - Q$$

$$\text{where } \left|\operatorname{Arg} S\right| \leq \frac{\pi}{2} - \gamma$$

$$\left|\operatorname{Arg} Q\right| \leq \frac{\pi}{2} - \gamma$$

and γ is a real vector of the same dimension as Z_2 satisfying $0 \leq \gamma < \frac{\pi}{2}$, with this substitution the above problem becomes

$$\text{minimize } \operatorname{Re}\left[C_1^* Z_1 + C_2^* S - C_2^*\right] \tag{3.1}$$

$$\text{subject to } \left|\operatorname{Arg}(A_1 Z_1 + A_2 S - A_2 Q - b)\right| \leq \beta \tag{3.2}$$

$$\left|\operatorname{Arg} Z_1\right| \leq \alpha \tag{3.3}$$

$$\left|\operatorname{Arg} S\right| \leq \frac{\pi}{2} - \gamma \tag{3.4}$$

$$\left|\operatorname{Arg} Q\right| \leq \frac{\pi}{2} - \gamma \tag{3.5}$$

The dual of this is

maximize $\quad \mathrm{Re}\ (b^*W)$ (3.6)

subject to $\left|\mathrm{Arg}\ (-A_1^*W + C_1)\right| \le \frac{\pi}{2} - \alpha$ (3.7)

$\left|\mathrm{Arg}\ (-A_2^*W + C_2)\right| \le \gamma$ (3.8)

$\left|\mathrm{Arg}\ (A_2^*W - C_2)\right| \le \gamma$ (3.9)

$\left|\mathrm{Arg}\ W\right| \le \frac{\pi}{2} - \beta$ (3.10)

Since $\gamma < \frac{\pi}{2}$ (3.8) and (3.9) imply $A_2^*W = C_2$

Now the above dual problem can be written as

D.P.(L) maximize $\quad \mathrm{Re}\ (b^*W)$ (3.11)

subject to $\left|\mathrm{Arg}\ (-A_1^*W + C_1)\right| \le \frac{\pi}{2} - \alpha$ (3.12)

$A_2^*W = C_2$ (3.13)

$\left|\mathrm{Arg}\ W\right| \le \frac{\pi}{2} - \beta$ (3.14)

<u>Complex Convex Programming Problem</u>.

P.P. (B) minimize $\quad F(Z) = \mathrm{Re}\left[f(Z)\right]$ (3.15)

subject to $\left|\mathrm{Arg}\ (AZ-b)\right| \le \beta$ (3.16)

$\left|\mathrm{Arg}\ Z\right| \le \alpha$ (3.17)

where A, b, α, β are as defined previously, $f(Z)=U(Z)+ iV(Z)$ is differentiable function of several complex variables. G_I is the region in n dimensional complex space C^n defined by vector Z satisfying (3.16) and (3.17). The function $\mathrm{Re}\left[f(Z)\right] = U(X,Y)$ is convex in G_I.

The dual problem of B is the following

D.P. ($\tilde{\tilde{B}}$)

$$\text{maximize} \quad G(\mu, W) = \mathrm{Re}\left[f(\mu) - \left(\overline{[f'(\mu)]}, \mu\right) + b^{*}W\right] \tag{3.18}$$

$$\text{subject to} \quad \left|\mathrm{Arg}\left(-A^{*}W + \overline{[f'(\mu)]}\right)\right| \leq \frac{\pi}{2} - \alpha \tag{3.19}$$

$$\left|\mathrm{Arg}\, W\right| \leq \frac{\pi}{2} - \beta \tag{3.20}$$

Where $f'(\mu)$ is a column vector i.e. $f'(\mu) = \left(\frac{\partial f}{\partial \mu_1}, \frac{\partial f}{\partial \mu_2}, \cdots \frac{\partial f}{\partial \mu_n}\right)^T$

Let C_{II} be the region defined in n dimensional complex space C^n by the vector μ satisfying (3.19) and (3.20).

$$\text{Define} \quad \varphi = f'(\mu) \tag{3.21}$$

D denotes the set of all vectors φ with $\mu \in C_{II}$. It will be assumed that there exists a differentiable function that determines for a given $\varphi_1 \in D$ a unique $\mu_1 \in C_{II}$ such that

$$\varphi_1 = [f'(\mu_1)]$$

Now if A, b have real entries and f(Z) is a real valued function defined in Euclidean space and if $\alpha = 0$, $\beta = \frac{\pi}{2}$ then the problem B and $\tilde{\tilde{B}}$ reduce to the dual convex programs in real space given by Dorn [25].

Lemma I If $f(Z) = U(X,Y) + i\,V(X,Y)$ is differentiable and $\mathrm{Re}\,[f(Z)] = U(X,Y)$ is convex function then for any two vectors Z and $\tilde{Z}$

$$\mathrm{Re}\left[f(Z) - f(\tilde{Z})\right] \geq \mathrm{Re}\left\{[f'(\tilde{Z})]^T (Z - \tilde{Z})\right\} = \mathrm{Re}\left(\overline{[f'(\tilde{Z})]}, (Z - \tilde{Z})\right)$$

Proof. $f(Z)$ is differentiable function therefore $U_X(X,Y) = V_Y(X,Y)$, $U_Y(X,Y) = -V_X(X,Y)$ (Cauchy Riemann conditions) and $U(X,Y)$ is convex we get

$$U(X,Y) - U(\breve{X},\breve{Y}) \geqslant \left[U_X(\breve{X},\breve{Y})\right]^T(X-\breve{X}) + \left[U_Y(\breve{X},\breve{Y})\right]^T(Y-\breve{Y})$$

$$= \left[U_X(\breve{X},\breve{Y})\right]^T(X-\breve{X}) - \left[V_X(\breve{X},\breve{Y})\right]^T(Y-\breve{Y})$$

$$= \operatorname{Re}\left\{\left[U_X(\breve{X},\breve{Y}) + i\, V_X(\breve{X},\breve{Y})\right]^T\left[(X-\breve{X}) + i(Y-\breve{Y})\right]\right\}$$

$$= \operatorname{Re}\left\{\left[f'(\breve{Z})\right]^T(Z-\breve{Z})\right\}$$

$$= \operatorname{Re}\left(\overline{\left[f'(\breve{Z})\right]}\,,\,(Z-\breve{Z})\right)$$

or $\quad R\left[f(Z) - f(\breve{Z})\right] \geqslant \operatorname{Re}\left\{\overline{\left[f'(\breve{Z})\right]}\,,\,(Z-\breve{Z})\right\}$.

<u>Theorem (3.A) (Weak Duality Theorem)</u>

If Z is any feasible solution to problem B and μ W is any feasible solution to problem $\breve{\bar{B}}$ then

$$\text{minimum} \quad \operatorname{Re}\left[f(Z)\right] \geqslant \text{maximum} \operatorname{Re}\left[f(\mu) - \left(\overline{\left[f'(\mu)\right]}, \mu\right) + b^*W\right]$$

<u>Proof</u>. For feasible Z, and μ , W we have from (3.17) and (3.19) that

$$\operatorname{Re}\left\{\left(-A^*W + \overline{\left[f'(\mu)\right]}\right)\,,\, Z\right\} \geqslant 0 \tag{3.22}$$

and from (3.16), (3.20)

$$\operatorname{Re}\left[W,\,(AZ-b)\right] \geqslant 0 \tag{3.23}$$

From (3.22) and (3.23)

$$\operatorname{Re}\left\{\left(\overline{\left[f'(\mu)\right]}\,,\,Z\right) - b^*W\right\} \geqslant 0 \tag{3.24}$$

using lemma I and (3.24)

$$\operatorname{Re}\left[f(Z) - f(\mu) - \left\{\overline{\left[f'(\mu)\right]}\,,\,(Z-\mu)\right\} + \left(\overline{\left[f'(\mu)\right]}\,,Z\right) - b^*W\right] \geqslant 0$$

$$\operatorname{Re}\left[f(Z)\right] \geqslant \operatorname{Re}\left[f(\mu) - \left(\overline{\left[f'(\mu)\right]}, \mu\right) + b^*W\right]$$

$$\text{minimum } R\left[f(Z)\right] \geqslant \text{maximum} \operatorname{Re}\left\{f(\mu) - \left(\overline{\left[f'(\mu)\right]}, \mu\right) + b^*W\right\} .$$

Hence proved.

Theorem (3.B) (Optimality Condition)

If the respective problem possess feasible $\tilde{Z}$, $\tilde{\mu}$, $\tilde{W}$ for which

$$\text{Re}\left[f(\tilde{Z})\right] = \text{Re}\left[f(\tilde{\mu}) - \left(\left[\overline{f'(\tilde{\mu})}\right], \tilde{\mu}\right) + b^*\tilde{W}\right] \tag{3.25}$$

then $\tilde{Z}$; $\tilde{\mu}$, $\tilde{W}$ are optimal feasible.

Proof. Let Z be any other feasible solution to B and $\tilde{\mu}$, $\tilde{W}$ is feasible to $\tilde{\tilde{B}}$ then from Weak Duality Theorem.

$$\text{Re}\left[f(Z)\right] \geqslant \text{Re}\left[f(\tilde{\mu}) - \left(\left[\overline{f'(\tilde{\mu})}\right], \tilde{\mu}\right) + b^*\tilde{W}\right]$$

$$= \text{Re}\left[f(\tilde{Z})\right] \quad \text{from (3.25)}$$

$$\text{Re}\left[f(Z)\right] \geqslant \text{Re}\left[f(\tilde{Z})\right]$$

Therefore $\tilde{Z}$ is optimal for B. A similar treatment shows that $\tilde{\mu}$, $\tilde{W}$ is optimal for $\tilde{\tilde{B}}$.

Hence Proved.

Theorem (3.C) Complementary Slackness Theorem.

$Z = Z_o$; $\mu = Z_o$, $W = W_o$ are optimal feasible solutions for respective problems iff (i) and (ii) are satisfied.

(i) $\quad \text{Re}\left[W_o, (AZ_o - b)\right] = 0$

(ii) $\quad \text{Re}\left[\left(-A^*W_o + \left[\overline{f'(Z_o)}\right]\right), Z_o\right] = 0.$

Proof. From primal and dual constraints we have for feasible solutions

$$\text{Re}\left[W_o, (AZ_o - b)\right] \geqslant 0 \tag{3.26}$$

$$\text{Re}\left[\left(-A^*W_o + \left[\overline{f'(Z_o)}\right]\right), Z_o\right] \geqslant 0 \tag{3.27}$$

Z_o; Z_o, W_o are optimal solutions therefore

$$\mathrm{Re}\left[f(Z_o)\right] = \mathrm{Re}\left[f(Z_o) - \left(\overline{(f'(Z_o))},\ Z_o\right) + b^*W_o\right]$$

$$\text{or}\quad \mathrm{Re}\left[\left(\overline{(f'(Z_o))},\ Z_o\right) - b^*W_o\right] = 0 \tag{3.28}$$

Adding (3.26) and (3.27)

$$\mathrm{Re}\left[W_o,\ (AZ_o - b)\right] + \mathrm{Re}\left[\left(-A^*W_o + \overline{\left[f'(Z_o)\right]}\ \right),\ Z_o\right] \geqslant 0 \tag{3.29}$$

$$\text{Since } (W_o,\ AZ_o) = (A^*W_o, Z_o) \tag{3.30}$$

(3.28), (3.30) and (3.29) give

$$\mathrm{Re}\left[W_o, (AZ_o - b)\right] + \mathrm{Re}\left[\left(-A^*W_o + \overline{\left[f'(Z_o)\right]}\ \right),\ Z_o\right] = 0 \tag{3.31}$$

But from (3.26) and (3.27) each term of (3.31) is nonnegative

therefore

$$\mathrm{Re}\left[W_o, (AZ_o - b)\right] = 0$$

$$\mathrm{Re}\left[\left(-A^*W_o + \overline{\left[f'(Z_o)\right]}\ \right),\ Z_o\right] = 0$$

Conversely if

$$\mathrm{Re}\left[W_o,\ (AZ_o - b)\right] = 0 \tag{3.32}$$

$$\text{and } \mathrm{Re}\left[\left(-A^*W_o + \overline{\left[f'(Z_o)\right]}\ \right),\ Z_o\right] = 0 \tag{3.33}$$

then Z_o; Z_o, W_o are optimal solutions.

Adding (3.32) and (3.33)

$$\mathrm{Re}\left[W_o,\ (AZ_o - b)\right] + \mathrm{Re}\left[\left(-A^*W_o + \overline{\left[f'(Z_o)\right]}\ \right),\ Z_o\right] = 0$$

using (3.30)

$$\mathrm{Re}\left[\left(\overline{\left[f'(Z_o)\right]}\ ,\ Z_o\right) - b^*W_o\right] = 0$$

or $\quad \operatorname{Re}\left[f(Z_o)\right] = \operatorname{Re}\left[f(Z_o) - \left(\overline{\left[f'(Z_o)\right]}, Z_o\right) + b^*W_o\right]$

which shows that Z_o; Z_o, W_o are optimal solutions.

Hence proved.

<u>Lemma 2.</u>

Consider the linear programming problem

P.P.(L) $\quad$ minimize $\quad \operatorname{Re}\left[-f(Z_o) + \left(\overline{\left[f'(Z_o)\right]}, Z\right)\right]$

subject to $\left|\operatorname{Arg}(AZ-b)\right| \leq \beta$

$$\left|\operatorname{Arg} Z\right| \leq \alpha$$

If Z_o is optimal solution to problem B then it is also optimal to linear programming problem L.

<u>Proof.</u>

The constraints of B and L are same therefore Z_o is also feasible to problem L,

and $\quad \operatorname{R}\left[f(Z_o)\right] \leq \operatorname{Re}\left[f(Z)\right] \quad (3.34)$

Suppose there exists Z_1 to this problem L with the property that

$$\operatorname{Re}\left[-f(Z_o) + \left(\overline{\left[f'(Z_o)\right]}, Z_1\right) + f(Z_o) - \left(\overline{\left[f'(Z_o)\right]}, Z_o\right)\right] < 0$$

$$\operatorname{Re}\left\{\overline{\left[f'(Z_o)\right]}, (Z_1 - Z_o)\right\} < 0 \quad (3.35)$$

Now $\quad Z_2 = Z_o + k\,(Z_1 - Z_o) \qquad 0 < k < 1$

is also feasible for both problems B and L.

Now from 2n dimensional mean value theorem for U(Z) and V(Z)

$$U(Z_2)-U(Z_o) = \nabla U\left[Z_o + k_1(Z_2-Z_o)\right] \cdot \left[Z_2-Z_o\right], \; 0 \leq k_1 \leq 1$$

$$V(Z_2)-V(Z_o) = \nabla V\left[Z_o + k_2(Z_2-Z_o)\right] \cdot \left[Z_2-Z_o\right], \; 0 \leq k_2 \leq 1$$

Since $f(Z)$ is differentiable and $\nabla U\,(Z) = \begin{pmatrix} U_X(Z) \\ U_Y(Z) \end{pmatrix}$

Therefore $\text{Re}\left[f(Z_2) - f(Z_o)\right] = \text{Re}\left[U(Z_2) + i\,V(Z_2) - U(Z_o) - i\,V(Z_o)\right]$

$$= \text{Re}\left[\left\{\nabla U\left[Z_o+k_1(Z_2-Z_o)\right] + i\nabla V\left[Z_o + k_2(Z_2-Z_o)\right]\right\} \cdot \left\{Z_2-Z_o\right\}\right]$$

Adding and substracting $\text{Re}\left(\overline{\left[f'(Z_o)\right]}\,,\,(Z_2-Z_o)\right)$

$$= \text{Re}\left(\left[\left\{\nabla U\left[Z_o+k_1(Z_2-Z_o)\right] + i\nabla V\left[Z_o+k_2(Z_2-Z_o)\right]\right\} \cdot \left\{Z_2-Z_o\right\}\right]\right.$$

$$\left. - \left(\overline{\left[f'(Z_o)\right]}\,,\,(Z_2-Z_o)\right) + \left(\overline{\left[f'(Z_o)\right]}\,,\,(Z_2-Z_o)\right]\right) \qquad (3.36)$$

Now

$$\left(\overline{\left[f'(Z_o)\right]}\,,\,(Z_2-Z_o)\right) = \left(f'(Z_o)^T(Z_2-Z_o)\right)$$

$$= \left[U_X(Z_o) + i\,V_X(Z_o)\right]^T\left[(X_2-X_o) + i\,(Y_2-Y_o)\right]$$

$$= \left\{\left[U_X(Z_o)\right]^T(X_2-X_o) - \left[V_X(Z_o)\right]^T(Y_2-Y_o)\right\} + i\left\{\left[U_X(Z_o)\right]^T(Y_2-Y_o)\right.$$

$$\left. + \left[V_X(Z_o)\right]^T(X_2-X_o)\right\}$$

Using Cauchy Riemann conditions

$$= \left\{\left[U_X(Z_o)\right]^T(X_2-X_o) + \left[U_Y(Z_o)\right]^T(Y_2-Y_o)\right\} + i\left\{\left[V_X(Z_o)\right]^T(X_2-X_o)\right.$$

$$\left. + \left[V_Y(Z_o)\right]^T(Y_2-Y_o)\right\}$$

$$= \left\{\nabla U(Z_o) \cdot (Z_2-Z_o) + i\,\nabla V(Z_o) \cdot (Z_2-Z_o)\right\} \qquad (3.37)$$

Substituting in (3.36) the value of $\left\{\overline{\left[f'(Z_o)\right]}\ ,\ (Z_2-Z_o)\right\}$ given by (3.37)

$$\mathrm{Re}\left[f(Z_2)-f(Z_o)\right] = \mathrm{Re}\ \Big(\ \Big\{\nabla U\left[Z_o+k_1(Z_2-Z_o)\right]+ i\nabla V\left[Z_o+k_2(Z_2-Z_o)\right]\Big\}\cdot\left\{Z_2-Z_o\right\}$$

$$-\Big\{\left[\nabla U(Z_o)+ i\nabla V(Z_o)\right]\cdot\left[Z_2-Z_o\right]\Big\}$$

$$+\ \left\{\overline{\left[f'(Z_o)\right]}\ ,\ (Z_2-Z_o)\right\}\Big)$$

using $Z_2 - Z_o = k(Z_1-Z_o)$ and rearranging

$$= k\ \mathrm{Re}\Big(\Big\{\nabla U\left[Z_o+ k_1k(Z_1-Z_o)\right] - \nabla U(Z_o)\Big\}\cdot\left\{Z_1-Z_o\right\}$$

$$+\ i\Big\{\nabla V\left[Z_o+ k_2k(Z_1-Z_o)\right] - \nabla V(Z_o)\Big\}\cdot\left\{Z_1-Z_o\right\}$$

$$+\ \left\{\overline{\left[f'(Z_o)\right]}\ ,\ (Z_1-Z_o)\right\}\Big) \qquad (3.38)$$

Since $\mathrm{Re}\left\{\overline{\left[f'(Z_o)\right]}\ ,\ (Z_1-Z_o)\right\}$ is independent of k and is negative, $\nabla U(Z)$, $\nabla V(Z)$ are continuous for k sufficiently small therefore we can choose k such that

$$\Big|\mathrm{Re}\ \Big(\Big\{\nabla U\ \left[Z_o+ k_1k(Z_1-Z_o)\right]\ - \nabla U(Z_o)\Big\}\cdot\left\{Z_1-Z_o\right\}$$

$$+\ i\Big\{\nabla V\left[Z_o+ k_2k\ (Z_1-Z_o)\right]\ -\ \nabla V(Z_o)\Big\}\cdot\left\{Z_1-Z_o\right\}\Big)\Big|$$

$$<\Big|\mathrm{Re}\Big\{\overline{\left[f'(Z_o)\right]}\ ,\ (Z_1-Z_o)\Big\}\Big|$$

Now using inequality (3.35)

$$\mathrm{Re}\left[f(Z_2) - f(Z_o)\right] < 0$$

This contradicts (3.34) therefore Z_o is also optimal for problem L.

Theorem (3.D) (Direct Duality Theorem)

If there exists a vector $Z = Z_o$ which minimizes $\operatorname{Re}\left[f(Z)\right]$ in problem B then there also exists vectors $\mu = Z_o$, $W = W_o$ which maximizes $G(\mu, W)$ in problem $\tilde{\tilde{B}}$ and moreover

$$\text{minimum } \operatorname{Re}\left[f(Z)\right] = \text{maximum } \operatorname{Re}\left[f(\mu) - \left(\overline{\left[f'(\mu)\right]}, \mu\right) + b^*W\right]$$

Proof. The dual of the linear programming problem L is

$$\text{D.P. } (\tilde{\tilde{L}}) \quad \text{maximize} \quad \operatorname{Re}\left[-f(Z_o) + b^*W\right]$$

$$\text{subject to} \quad \left|\operatorname{Arg}\left(-A^*W + \overline{\left[f'(Z_o)\right]}\right)\right| \leq \frac{\pi}{2} - \alpha$$

$$\left|\operatorname{Arg} W\right| \leq \frac{\pi}{2} - \beta$$

Let $W = W_o$ maximizes $\tilde{\tilde{L}}$ therefore from the duality theory of linear programming in complex space given by Levinson (59) we have

$$\operatorname{Re}\left[-f(Z_o) + \left(\overline{\left[f'(Z_o)\right]}, Z_o\right)\right] = \operatorname{Re}\left[-f(Z_o) + b^*W_o\right]$$

$$\text{or} \qquad \operatorname{Re}\left(\overline{\left[f'(Z_o)\right]}, Z_o\right) = \operatorname{Re}\left(b^*W_o\right) \tag{3.39}$$

Now $\mu = Z_o$ and $W = W_o$ is a feasible solution to problem $\tilde{\tilde{B}}$, consider any other feasible solution μ, W then

$$G(Z_o, W_o) - G(\mu, W) = \operatorname{Re}\Big[f(Z_o) - \left(\overline{\left[f'(Z_o)\right]}, Z_o\right) + b^*W_o - f(\mu)$$

$$+ \left(\overline{\left[f'(\mu)\right]}, \mu\right) - b^*W\Big]$$

using (3.39)

$$G(Z_o, W_o) - G(\mu, W) = \operatorname{Re}\left[f(Z_o) - f(\mu) + \left(\overline{\left[f'(\mu)\right]}, \mu\right) - b^*W\right]$$

$$\geqslant \operatorname{Re}\left[\left\{\overline{\left[f'(\mu)\right]}, (Z_o - \mu)\right\} + \left(\overline{f'(\mu)}, \mu\right) - b^*W\right]$$

(by lemma I)

$$G(Z_o,W_o)-G(\mu ,W) \geqslant \mathrm{Re}\left\{ (\overline{[f'(\mu)]} , Z_o) - b^*W \right\} \qquad (3.40)$$

For feasible Z_o; μ , W we have from (3.17) and (3.19)

$$\mathrm{Re} \left[(-A^*W + \overline{[f'(\mu)]}), Z_o\right] \geqslant 0 \qquad (3.41)$$

Similarly from (3.16) and (3.20)

$$\mathrm{Re} \left[W,(AZ_o-b)\right] \geqslant 0 \qquad (3.42)$$

From (3.41) and (3.42)

$$\mathrm{Re}\left\{ (\overline{[f'(\mu)]} , Z_o) - b^*W\right\} \geqslant 0 \qquad (3.43)$$

using (3.43) in (3.40)

$$G(Z_o,W_o) - G(\mu ,W) \geqslant 0$$

Thus (Z_o,W_o) is a maximizing solution to problem $\tilde{\tilde{B}}$.

Now $\quad G(Z_o,W_o) = \mathrm{Re} \left[f(Z_o) - (\overline{[f'(Z_o)]}, Z_o) + b^*W_o\right]$

using (3.39)

$$= \mathrm{Re} \left[f(Z_o)\right]$$

which varifies the equality of the objective function.

Hence proved.

<u>Lemma 3.</u> If $(\mu_0 , W_o, \varsigma_0)$ is a maximizing solution of problem $\tilde{\tilde{B}}$ which on repeasing can be read as

$$\text{maximize} \quad G(\mu ,W, \varsigma) = \mathrm{Re}\left[f(\mu)- \varsigma^T\mu + b^*W \right] \qquad (3.44)$$

$$\text{subject to} \quad \left|\mathrm{Arg}\ (-A^*W + \overline{\varsigma})\right| \leq \frac{\pi}{2} - \alpha \qquad (3.45)$$

$$\left|\mathrm{Arg}\ W\right| \leq \frac{\pi}{2} - \beta \qquad (3.46)$$

$$\varsigma = f'(\mu) \qquad (3.47)$$

then (W_o, ς_0) maximizes the linear programming problem

$$(L_2) \quad \text{maximize} \quad H(W, \varsigma) = \operatorname{Re}\left[f(\mu_0) - \varsigma^T \mu_0 + b^* W\right] \tag{3.48}$$

$$\text{subject to} \quad \left|\operatorname{Arg}\left(-A^* W + \overline{\varsigma}\right)\right| \leq \frac{\pi}{2} - \alpha \tag{3.49}$$

$$\left|\operatorname{Arg} W\right| \leq \frac{\pi}{2} - \beta \tag{3.50}$$

<u>Proof</u>. $(\mu_0, W_o, \varsigma_0)$ is a solution to problem $\tilde{\tilde{S}}$ therefore $G(\mu_0, W_o, \varsigma_0) \geqslant G(\mu, W, \varsigma)$ for all μ, W, ς satisfying the constraints.

From constraints (3.45), (3.46) and (3.49), (3.50) we conclude that $W = W_o$, $\varsigma = \varsigma_0$ is a feasible solution to problem L_2. Suppose there exists W_2, ς_2 satisfying (3.49) and (3.50) such that

$$\operatorname{Re}\left[f(\mu_0) - \varsigma_2^T \mu_0 + b^* W_2\right] > \operatorname{Re}\left[f(\mu_0) - \varsigma_0^T \mu_0 + b^* W_o\right]$$

$$\text{or} \quad \operatorname{Re}\left[\varsigma_0^T \mu_0 - \varsigma_2^T \mu_0 + b^* W_2 - b^* W_o\right] > 0 \tag{3.51}$$

Now define
$$W_1 = W_o + k\,(W_2 - W_o) \tag{3.52}$$

$$\varsigma_1 = \varsigma_0 + k\,(\varsigma_2 - \varsigma_0) \tag{3.53}$$

For $0 < k \leq 1$, then W_1, ς_1 is also feasible solution to problem L_2.

Since $\varsigma_1 \in D$ there exists a μ_1 in C_{II} such that

$$\varsigma_1 = [f'(\mu_1)] \qquad (\varsigma = \xi + i\eta)$$

Therefore μ_1, W_1, ς_1 is a feasible solution to problem $\tilde{\tilde{S}}$.

Consider

$$G(\mu_1, W_1, \varsigma_1) - G(\mu_0, W_0, \varsigma_0) = Re\left[f(\mu_1) - \varsigma_1^T \mu_1 + b^* W_1 - f(\mu_0) + \varsigma_0^T \mu_0 - b^* W_0\right]$$

$$= Re\left[U(\mu_1) + iV(\mu_1) - U(\mu_0) - iV(\mu_0) - \varsigma_1^T \mu_1 + \varsigma_0^T \mu_0 + b^* W_1 - b^* W_0\right]$$

By 2n dimensional mean value theorem for $U(\mu)$ and $V(\mu)$

$$= Re\Big[\left(\nabla U[\mu_1 - k_1(\mu_1 - \mu_0)] + i\nabla V[\mu_1 - k_2(\mu_1 - \mu_0)]\right) \cdot$$

$$\cdot (\mu_1 - \mu_0) - \varsigma_1^T \mu_1 + \varsigma_0^T \mu_0 + b^* W_1 - b^* W_0\Big]$$

Adding and subestracting $Re\left[\nabla U(\mu_1) + i\nabla V(\mu_1)\right] \cdot [\mu_1 - \mu_0]$

$$= Re\Big\{\Big[\left(\nabla U[\mu_1 - k_1(\mu_1 - \mu_0)] - \nabla U(\mu_1)\right) + i\left(\nabla V[\mu_1 - k_2(\mu_1 - \mu_0)]\right.$$

$$\left. - \nabla V(\mu_1)\right)\Big] \cdot [\mu_1 - \mu_0]$$

$$+ \left[\nabla U(\mu_1) + i\nabla V(\mu_1)\right] \cdot [\mu_1 - \mu_0] - \varsigma_1^T \mu_1 + \varsigma_0^T \mu_0 + b^* W_1 - b^* W_0\Big\}$$

(3.54)

Now $\left[\nabla U(\mu_1) + i\nabla V(\mu_1)\right] \cdot [\mu_1 - \mu_0] = \left[U_d(\mu_1) + iV_d(\mu_1)\right] \cdot [d_1 - d_0]$

$$+ \left[U_e(\mu_1) + iV_e(\mu_1)\right] \cdot [e_1 - e_0]$$

$$= \left[U_d(\mu_1) + i\; V_d(\mu_1)\right] \cdot [d_1 - d_0] + i\; \left[U_d(\mu_1) + iV_d(\mu_1)\right] \cdot [e_1 - e_0]$$

(using Cauchy Riemann conditions i.e.

$U_d(\mu_1) = V_e(\mu_1)$, $U_e(\mu_1) = -V_d(\mu_1)$.).

$$= \left[U_d(\mu_1) + iV_d(\mu_1)\right]^T \left[(d_1 - d_0) + i(e_1 - e_0)\right]$$

$$= \left[U_d(\mu_1) + iV_d(\mu_1)\right]^T [\mu_1 - \mu_0] = \left[f'(\mu_1)\right]^T [\mu_1 - \mu_0]$$

$$= \varsigma_1^T (\mu_1 - \mu_0)$$

or $$\left[\nabla U(\mu_1) + i\nabla V(\mu_1)\right]\cdot\left[\mu_1 - \mu_0\right] = \rho_1^T\mu_1 - \rho_1^T\mu_0 \tag{3.55}$$

Substituting the value of $\left[\nabla U(\mu_1) + i\nabla V(\mu_1)\right]\cdot\left[\mu_1 - \mu_0\right]$ given by (3.55) in (3.54)

$$G(\mu_1, W_1, \rho_1) - G(\mu_0, W_0, \rho_0) = \operatorname{Re}\Big(\Big[\big\{\nabla U[\mu_1 - k_1(\mu_1 - \mu_0)] - \nabla U(\mu_1)\big\}$$
$$+ i\big\{\nabla V[\mu_1 - k_2(\mu_1 - \mu_0)] - \nabla V(\mu_1)\big\}\Big]\cdot\left[\mu_1 - \mu_0\right] - (\rho_1 - \rho_0)^T\mu_0 +$$
$$+ b^*(W_1 - W_0)\Big)$$

$$= \operatorname{Re}\Big(\Big[\big\{\nabla U[\mu_1 - k_1(\mu_1 - \mu_0)] - \nabla U(\mu_1)\big\} + i\big\{\nabla V[\mu_1 - k_2(\mu_1 - \mu_0)]$$
$$- \nabla V(\mu_1)\big\}\Big]^T \begin{bmatrix} d_1 - d_0 \\ e_1 - e_0 \end{bmatrix}$$

$$- (\rho_1 - \rho_0)^T\mu_0 + b^*(W_1 - W_0)\Big) \tag{3.56}$$

Now $\rho = f'(\mu)$ and as the inverse of f' exists

$$\mu = (f')^{-1}(\rho) = \phi(\rho) + i\,\psi(\rho).$$

By assumption the inverse gradient of f has a derivative therefore

$$\mu_1 - \mu_0 = (f')^{-1}(\rho_1) - (f')^{-1}(\rho_0) = \left[\phi(\rho_1) - \phi(\rho_0)\right] + i\left[\psi(\rho_1) - \psi(\rho_0)\right]$$

or

$$\begin{bmatrix} d_1^1 - d_0^1 \\ d_1^2 - d_0^2 \\ \vdots \\ d_1^n - d_0^n \\ e_1^1 - e_0^1 \\ e_1^2 - e_0^2 \\ \vdots \\ e_1^n - e_0^n \end{bmatrix} = \begin{bmatrix} \frac{\partial\phi_1}{\partial\xi^1} & \frac{\partial\phi_1}{\partial\xi^2} & \cdots & \frac{\partial\phi_1}{\partial\xi^n} & \frac{\partial\phi_1}{\partial\eta^1} & \frac{\partial\phi_1}{\partial\eta^2} & \cdots & \frac{\partial\phi_1}{\partial\eta^n} \\ \frac{\partial\phi_2}{\partial\xi^1} & \frac{\partial\phi_2}{\partial\xi^2} & \cdots & \frac{\partial\phi_2}{\partial\xi^n} & \frac{\partial\phi_2}{\partial\eta^1} & \frac{\partial\phi_2}{\partial\eta^2} & \cdots & \frac{\partial\phi_2}{\partial\eta^n} \\ \vdots & & & & & & & \\ \frac{\partial\phi_n}{\partial\xi^1} & \frac{\partial\phi_n}{\partial\xi^2} & \cdots & \frac{\partial\phi_n}{\partial\xi^n} & \frac{\partial\phi_n}{\partial\eta^1} & \frac{\partial\phi_n}{\partial\eta^2} & \cdots & \frac{\partial\phi_n}{\partial\eta^n} \\ \frac{\partial\psi_1}{\partial\xi^1} & \frac{\partial\psi_1}{\partial\xi^2} & \cdots & \frac{\partial\psi_1}{\partial\xi^n} & \frac{\partial\psi_1}{\partial\eta^1} & \frac{\partial\psi_1}{\partial\eta^2} & \cdots & \frac{\partial\psi_1}{\partial\eta^n} \\ \frac{\partial\psi_2}{\partial\xi^1} & \frac{\partial\psi_2}{\partial\xi^2} & \cdots & \frac{\partial\psi_2}{\partial\xi^n} & \frac{\partial\psi_2}{\partial\eta^1} & \frac{\partial\psi_2}{\partial\eta^2} & \cdots & \frac{\partial\psi_2}{\partial\eta^n} \\ \vdots & & & & & & & \\ \frac{\partial\psi_n}{\partial\xi^1} & \frac{\partial\psi_n}{\partial\xi^2} & \cdots & \frac{\partial\psi_n}{\partial\xi^n} & \frac{\partial\psi_n}{\partial\eta^1} & \frac{\partial\psi_n}{\partial\eta^2} & \cdots & \frac{\partial\psi_n}{\partial\eta^n} \end{bmatrix} \begin{bmatrix} \xi_1^1 - \xi_0^1 \\ \xi_1^2 - \xi_0^2 \\ \vdots \\ \xi_1^n - \xi_0^n \\ \eta_1^1 - \eta_0^1 \\ \eta_1^2 - \eta_0^2 \\ \vdots \\ \eta_1^n - \eta_0^n \end{bmatrix}$$

$$\begin{pmatrix} d_1 - d_0 \\ e_1 - e_0 \end{pmatrix} = M\left[\left\{ \zeta_0 - k_3(\zeta_1 - \zeta_0)\right\}, \left\{\zeta_0 - k_4(\zeta_1 - \zeta_0)\right\}\right]\begin{bmatrix} \xi_1 - \xi_0 \\ \eta_1 - \eta_0 \end{bmatrix} \qquad (3.57)$$

or $$(\mu_1 - \mu_0) = M\left[\left\{\zeta_0 - k_3(\zeta_1 - \zeta_0)\right\}, \left\{\zeta_0 - k_4(\zeta_1 - \zeta_0)\right\}\right]\left[\zeta_1 - \zeta_0\right]$$

where ϕ and ψ are evaluated at $[\zeta_0 - k_3(\zeta_1 - \zeta_0)]$, $[\zeta_0 - k_4(\zeta_1 - \zeta_0)]$ respectively.

Substituting the value of $\begin{pmatrix} d_1 - d_0 \\ e_1 - e_0 \end{pmatrix}$ given by (3.57) in (3.56) and using (3.52), (3.53)

$$G(\mu_1, W_1, \zeta_1) - G(\mu_0, W_0, \zeta_0) = k\,Re\Big(\Big[\Big\{\nabla U[\mu_1 - k_1 k(\mu_2 - \mu_0)] - \nabla U(\mu_1)\Big\}$$

$$+ i\Big\{\nabla V[\mu_1 - k_2 k(\mu_2 - \mu_0)] - \nabla V(\mu_1)\Big\}\Big] \cdot M\Big[\Big\{\zeta_0 - k_3 k(\zeta_2 - \zeta_0)\Big\},$$

$$\Big\{\zeta_0 - k_4 k(\zeta_2 - \zeta_0)\Big\}\Big]\begin{pmatrix} \xi_2 - \xi_0 \\ \eta_2 - \eta_0 \end{pmatrix}$$

$$- (\zeta_2 - \zeta_0)^2 \mu_0 + b^*(W_2 - W_0)\Big) \qquad (3.58)$$

Since the last two terms are independent of k and $\nabla U(\mu)$ $\nabla V(\mu)$ are continuous therefore by choosing k sufficiently small the following inequality can be satisfied

$$\Big| Re\Big(\Big[\Big\{\nabla U[\mu_1 - k_1 k(\mu_2 - \mu_0)] - \nabla U(\mu_1)\Big\} + i\Big\{\nabla V[\mu_1 - k_2 k(\mu_2 - \mu_0)] - \nabla(\mu_1)\Big\}\Big]$$

$$M\Big[\Big\{\zeta_0 - k_3 k(\zeta_2 - \zeta_0)\Big\}, \Big\{\zeta_0 - k_4 k(\zeta_2 - \zeta_0)\Big\}\Big]\begin{bmatrix} \xi_2 - \xi_0 \\ \eta_2 - \eta_0 \end{bmatrix}\Big|$$

$$< \Big| Re\,\Big[-(\zeta_2 - \zeta_0)^2 \mu_0 + b^*(W_2 - W_0)\Big]\Big| \qquad (3.59)$$

The term of the right hand side of (3.59) inside absolute value is positive by (3.51) therefore by taking k sufficiently small (3.58)

can be made positive .

Thus $G(\mu_1, W_1, \varsigma_1) - G(\mu_0, W_0, \varsigma_0) > 0$.

This contradicts the hypothesis that $(\mu_0, W_0, \varsigma_0)$ is a maximizing solution to problem $\tilde{\tilde{B}}$. It follows that (W_0, ς_0) maximizes L_2.

Hence the proof.

Theorem (3.E) (Converse Duality Theorem)

If μ_0, W_0 are vectors which maximizes $G(\mu, W)$ in problem $\tilde{\tilde{B}}$ then $Z = \mu_0$ minimizes $\mathrm{Re}\,[f(Z)]$ in problem B and moreover

minimum $\mathrm{Re}\,[f(Z)]$ = maximum $\mathrm{Re}\,[f(\mu) - (\overline{[f'(\mu)]}, \mu) + b^*W]$

Proof. The dual of the linear programming problem L_2, which is of the type (L) according to $(\tilde{L})$, is

$$\text{D.P. } (\tilde{\tilde{L}}_2) \quad \text{minimize} \quad F(Z) = \mathrm{Re}\,[f(\mu_0)] \tag{3.60}$$

$$\text{subject to } \left|\mathrm{Arg}\,(AZ-b)\right| \leq \beta \tag{3.61}$$

$$\left|\mathrm{Arg}\, Z\right| \leq \alpha \tag{3.62}$$

$$Z = \mu_0 \tag{3.63}$$

From lemma 3, P.P.(L_2) has a solution therefore the duality theorem of linear programming assures the existense of a solution to this problem. Now by (3.63) $Z = \mu_0$ is that solution. Therefore from duality theorem

$$\mathrm{Re}\,[f(\mu_0)] = \mathrm{Re}\,[f(\mu_0) - \varsigma_0^T \mu_0 + b^* W_0]$$

$$\text{or} \quad \mathrm{Re}\,[\varsigma_0^T \mu_0 - b^* W_0] = 0 \tag{3.64}$$

Since μ_0 satisfies (3.61) and (3.62) therefore it is feasible for

problem B. Let Z be any other feasible solution then for lemma I

$$\mathrm{Re}\left[f(Z) - f(\mu_0)\right] \geqslant \quad \mathrm{Re}\left\{\left[\overline{f'(\mu_0)}\right], (Z - \mu_0)\right\} \tag{3.65}$$

From (3.47) and (3.64) $\quad \mathrm{Re}\left\{ \left[f'(\mu)\right]^T \mu_0 - b^* W_0 \right\} = 0$

or $\quad \mathrm{Re}\left\{f'(\mu_0)^T \mu_0\right\} = \mathrm{Re}\,(b^* W_0)$ (3.66)

From (3.16) and (3.20)

$$\mathrm{Re}\left[W_0, (AZ - b)\right] \geqslant 0 \tag{3.67}$$

Similarly from (3.17) and (3.19)

$$\mathrm{Re}\left[\left(-A^* W_0 + \left[\overline{f'(\mu_0)}\right]\right), Z\right] \geqslant 0 \tag{3.68}$$

From (3.67) and (3.68)

$$\mathrm{Re}\left\{\left(\left[\overline{f'(\mu_0)}\right], Z\right) - b^* W_0\right\} \geqslant 0$$

using (3.66)

$$\mathrm{Re}\left\{\left(\left[\overline{f'(\mu_0)}\right], Z\right)\right\} - \mathrm{Re}\left\{\left[f'(\mu_0)\right]^T \mu_0\right\} \geqslant 0$$

$$\text{or} \quad \mathrm{Re}\left\{\left[\overline{f'(\mu_0)}\right], (Z - \mu_0)\right\} \geqslant 0 \tag{3.69}$$

using (3.69) in (3.65)

$$\mathrm{Re}\left[f(Z) - f(\mu_0)\right] \geqslant 0$$

$$\mathrm{Re}\left[f(Z)\right] \geqslant \mathrm{Re}\left[f(\mu_0)\right].$$

Thus μ_0 minimises B

Now from (3.64) and (3.47)

$$\mathrm{Re}\left[f(\mu_0)\right] = \mathrm{Re}\left[f(\mu_0) - \left(\left[\overline{f'(\mu_0)}\right], \mu_0\right) + b^* W_0\right] = G(\mu_0, W_0)$$

which shows the equality of objective functions.

Hence proved.

CHAPTER - IV

Symmetric Dual and Self Dual Nonlinear Programming Problems in Complex Space

The purpose of this chapter is to present the symmetric dual and self dual nonlinear programming problems in complex space. The duality theorem for symmetric programs is proved by adopting the technique of Dorn [25] and Mond [66], and using the duality theorem for linear programming in complex space [59].

The chapter is divided into two sections. Section I deals with symmetric dual programs in complex space. Duality Theorem, Weak Duality Theorem, Optimality Condition and Complementary Slackness Theorem are proved. Duality theorem

assures that if primal problem is solvable at certain point then dual problem is also solvable at the same point, conversly if dual problem is solvable at certain point, then the primal problem is also solvable at the same point.

Weak duality theorem is proved under the assumption that $Re\ [f(Z)] = U(X,Y)$ and $Re\ [g(W)] = p(\gamma, \delta)$ are convex functions in the regions defined by vectors $X + i\ Y = Z$ and $\gamma + i\ \delta = W$ satisfying the primal constraints. Duality theorem requires in addition that the inverse gradient of functions f and g have derivatives, i.e. the matrix M_1 consisting the partial derivatives of J and N with respect to ξ, η; and the matrix M_2 consisting the partial derivatives of p and q with respect to s and t exist.

Section II is devoted to the study of symmetric dual and self dual programs. Weak duality theorem is proved under the assumption that $Re\ [K(Z,W)]$ is convex concave function and $K(Z,W)$ is differentiable. The symmetric dual program is reduced to a self dual program under the assumption that $K(Z,W)$ is skew symmetric i.e. $K(Z,W) = -K(W,Z)$, and the self duality theorem is proved. Composite program is obtained from the pair of symmetric dual nonlinear programs in complex space.

SECTION - I

General Symmetric Dual Programs in Complex Space

The symmetric dual programs in complex space may be stated as

P.P. (E)

$$\text{minimize} \quad F(Z,W) = Re \left\{ [g'(W)]^T W - g(W) + f(Z) \right\} \qquad (4.1.1)$$

$$\text{subject to} \quad \left| \operatorname{Arg} \left(AZ + \overline{[g'(W)]} - b \right) \right| \leq \beta \tag{4.1.2}$$

$$\left| \operatorname{Arg} Z \right| \leq \alpha \tag{4.1.3}$$

D.P. $(\tilde{\tilde{B}})$

$$\text{maximize} \quad G(\lambda, \mu) = \operatorname{Re} \left[-g(\mu) + f(\lambda) - [f'(\lambda)]^T \lambda + b^* \mu \right] \tag{4.1.4}$$

$$\text{subject to} \quad \left| \operatorname{Arg} \left(-A^* \mu + \overline{[f'(\lambda)]} \right) \right| \leq \frac{\pi}{2} - \alpha \tag{4.1.5}$$

$$\left| \operatorname{Arg} \mu \right| \leq \frac{\pi}{2} - \beta \tag{4.1.6}$$

C_I and C_{II} respectively denote the regions in n and m dimensional complex spaces defined by vectors Z and W satisfying the constraints (4.1.2) and (4.1.3). The function Re $[f(Z)]$ = $U(X,Y)$ is convex in C_I, and Re $[g(W)]$ = $p(\gamma, \delta)$ is convex in C_{II}. Similarly λ and μ are n- and m- vectors respectively. Let C_{III} and C_{IV} respectively denote the regions in n and m dimensional complex space defined by vectors λ and μ satisfying the constraints (4.1.5) and (4.1.6). The function Re $[f(\lambda)]$ = $U(a,c)$ is convex in C_{III}, and the function Re $[g(\mu)]$ = $P(d,e)$ is convex in C_{IV}.

__Theorem (4.1A) (Weak Duality Theorem).__ If (Z,W) and (λ, μ) are feasible solutions to problem B and $\tilde{\tilde{B}}$ respectively, then minimum

$$\operatorname{Re} \left[[g'(W)]^T W - g(W) + f(Z) \right]$$

$$\geq \text{maximum} \quad \operatorname{Re} \left[-g(\mu) + f(\lambda) - [f'(\lambda)]^T \lambda + b^* \mu \right].$$

__Proof.__ (Z,W) is feasible for problem B and (λ, μ) is feasible for problem $\tilde{\tilde{B}}$, therefore from (4.1.2), (4.1.6) and (4.1.5) (4.1.3)

$$\mathrm{Re}\left[\mu\ ,\ (AZ + \overline{[g'(w)]} - b)\right] \geqslant 0 \qquad (4.1.7)$$

$$\mathrm{Re}\left[(-A^{*}\mu + \overline{[f'(\lambda)]}\ ),\ z\right] \geqslant 0 \qquad (4.1.8)$$

Since $$(\mu,\ AZ) = (A^{*}\mu\ ,z), \qquad (4.1.9)$$

using (4.1.7) (4.1.8) and (4.1.9),

$$\mathrm{Re}\left\{[\mu\ ,\overline{(g'(w))}] - b^{*}\mu + [f'(\lambda)]^{T} z\right\} \geqslant 0$$

or $$\mathrm{Re}\left\{[g'(w)]^{T}\mu + [f'(\lambda)]^{T} z - b^{*}\mu\right\} \geqslant 0. \qquad (4.1.10)$$

The convexity of $\mathrm{Re}\,[f(z)]$ with respect to X,Y and $\mathrm{Re}\,[g(w)]$ with respect to γ, δ gives

$$\mathrm{Re}\left[f(z) - f(\lambda) - [f'(\lambda)]^{T}(z-\lambda)\right] \geqslant 0, \qquad (4.1.11)$$

and $$\mathrm{Re}\left[g(\mu) - g(w) - [g'(w)]^{T}(\mu - w)\right] \geqslant 0 \qquad (4.1.12)$$

Now using (4.1.10) (4.1.11) and (4.1.12)

$$\mathrm{Re}\Big\{[g'(w)]^{T}\mu + [f'(\lambda)]^{T} z - b^{*}\mu + f(z) - f(\lambda) - [f'(\lambda)]^{T}(z-\lambda)$$
$$+ g(\mu) - g(w) - [g'(w)]^{T}(\mu - w)\Big\} \geqslant 0,$$

i.e., $$\mathrm{Re}\left\{f(z) - f(\lambda) + [f'(\lambda)]^{T}\lambda + g(\mu) - g(w) + [g'(w)]^{T} w - b^{*}\mu\right\} \geqslant 0,$$

i.e., $$\mathrm{Re}\left\{[g'(w)]^{T} w - g(w) + f(z)\right\} \geqslant \mathrm{Re}\Big[-g(\mu) + f(\lambda) - [f'(\lambda)]^{T}\lambda$$
$$+ b^{*}\mu\Big],$$

or $$\text{minimum}\ \mathrm{Re}\left[[g'(w)]^{T} w - g(w) + f(z)\right]$$
$$\text{maximum}\ \mathrm{Re}\left[-g(\mu) + f(\lambda) - [f'(\lambda)]^{T}\lambda + b^{*}\mu\right].$$

Lemma I. If (Z_o, W_o, ζ_o) is a minimising solution of problem B, which on rephasing can be read as,

$$\text{minimise} \quad F(Z,W,\zeta) = \text{Re}\left[\zeta^T W - g(W) + f(Z)\right], \tag{4.1.13}$$

$$\text{subject to} \quad \left|\text{Arg}\,(AZ + \overline{\zeta} - b)\right| \leq \beta \tag{4.1.14}$$

$$\left|\text{Arg}\, Z\right| \leq \alpha \tag{4.1.15}$$

$$\zeta = g'(W), \tag{4.1.16}$$

then (Z_o, ζ_o) minimises the linear programming problem,

P.P. (L_2)

$$\text{minimise} \quad H(Z,\zeta) = \text{Re}\left[\zeta^T W_o + \left[f'(Z_o)\right]^T Z\right], \tag{4.1.17}$$

$$\text{subject to} \quad \left|\text{Arg}\,(AZ + \overline{\zeta} - b)\right| \leq \beta \tag{4.1.18}$$

$$\left|\text{Arg}\, Z\right| \leq \alpha \tag{4.1.19}$$

Proof. We denote by D the domain of values of $\zeta = \xi + i\eta$ in C^m for which (4.1.16) holds.

(Z_o, W_o, ζ_o) is a solution of problem B therefore $F(Z_o, W_o, \zeta_o) - F(Z,W,\zeta) \leq 0$ for all Z, W, ζ satisfying the constraints of problem B. Since the constraints (4.1.14), (4.1.15) and (4.1.18), (4.1.19) are same therefore (Z_o, ζ_o) is also feasible solution to L_2. Suppose that (Z_1, ζ_1) be another feasible solution to L_2 such that,

$$H(Z_1, \zeta_1) < H(Z_o, \zeta_o)$$

i.e. $\mathrm{Re}\left\{ \varsigma_1^T w_o + [f'(z_o)]^T z_1 - \varsigma_o^T w_o - [f(z_o)]^T z_o \right\} < 0$ (4.1.20)

Define $z_2 = z_o + k\,(z_1 - z_o),$ (4.1.21)

$\varsigma_2 = \varsigma_o + k\,(\varsigma_1 - \varsigma_o),$ (4.1.22)

for $0 < k \leq 1$. By lemma (2) of [42] z_2, ς_2 is feasible for problems B and L_2, therefore $\varsigma_2 \in D$.

Since $\varsigma_2 \in D$ there exists a vector $w_2 \in C_{II}$ such that

$$\varsigma_2 = g'(w_2).$$

Thus (z_2, w_2, ς_2) is a feasible solution to problem B.

Consider

$$F(z_2, w_2, \varsigma_2) - F(z_o, w_o, \varsigma_o)$$

$$= \mathrm{Re}\left[[g'(w_2)]^T w_2 - g(w_2) + f(z_2) - [g'(w_o)]^T w_o + g(w_o) - f(z_o) \right]$$

$$= \mathrm{Re}\left[-\{g(w_2) - g(w_o)\} + \{f(z_2) - f(z_o)\} + \varsigma_2^T w_2 - \varsigma_o^T w_o \right]$$

Using 2m and 2n dimensional mean value theorem for $p(w)$, $q(w)$ and $U(z)$, $V(z)$, and adding substracting $\mathrm{Re}\{\nabla_1 p(w_2) + i\, \nabla_1 q(w_2)\} \cdot \{w_2 - w_o\}$

and $\mathrm{Re}\{\nabla U(z_o) + i\, \nabla V(z_o)\} \cdot \{z_2 - z_o\}$

$$F(z_2, w_2, \varsigma_2) - F(z_o, w_o, \varsigma_o) = \mathrm{Re}\Big[-\{\nabla_1 p[w_2 - \theta_1(w_2 - w_o)] + i\, \nabla_1 q[w_2 - \theta_1(w_2 - w_o)]\} \cdot \{w_2 - w_o\}$$

$$+ \left\{\nabla_1 p(w_2) + i\ \nabla_1 q(w_2)\right\}\cdot\left\{w_2 - w_0\right\} - \left\{\nabla_1 p(w_2) + i\,\nabla_1 q(w_2)\right\}\cdot\left\{w_2 - w_0\right\}$$

$$+ \left\{\nabla\ U\left[z_2 - \theta_2(z_2 - z_0)\right] + i\nabla V\left[z_2 - \theta_2(z_2 - z_0)\right]\right\}\cdot\left\{z_2 - z_0\right\}$$

$$+ \left\{\nabla U(z_0) + i\nabla V(z_0)\right\}\cdot\left\{z_2 - z_0\right\} - \left\{\nabla U(z_0) + i\nabla V(z_0)\right\}\cdot\left\{z_2 - z_0\right\}$$

$$+ \quad \xi_2^T w_2 - \xi_0^T w_0\Big]$$

Taking $\theta_2 = 1 - \theta_1$ and rearranging the terms

$F(z_2, w_2, \xi_2) - F(z_0, w_0, \xi_0)$

$$= \operatorname{Re}\Big[-\Big(\Big\{\nabla_1 p\left[w_2 - \theta_1(w_2 - w_0)\right] - \nabla_1 p(w_2)\Big\} + i\Big\{\nabla_1 q\left[w_2 - \right.$$

$$\left. - \theta(w_2 - w_0)\right] - \nabla_1 q(w)\Big\}$$

$$\cdot\left\{w_2 - w_0\right\}\Big) + \Big(\Big\{\nabla\ U\left[z_0 + \theta_1(z_2 - z_0)\right] - \nabla U(z_0)\Big\} + i\Big\{\nabla V\left[z_0 + \theta_1(z_2 - z_0)\right]$$

$$+ i\nabla V(z_0)\Big\}\Big)\cdot\left(z_2 - z_0\right) - \left\{\nabla_1 p(w_2) + i\ \nabla_1 q(w_2)\right\}\cdot\left\{w_2 - w_0\right\}$$

$$+ \ \xi_2^T w_2 - \xi_0^T w_0 + \left\{\nabla U(z_0) + i\ \nabla V(z_0)\right\}\cdot\left\{z_2 - z_0\right\} \qquad (4.1.23)$$

Now $\quad \left[\nabla_1 p(w_2) + i\ \nabla_1 q(w_2)\right]\cdot\left[w_2 - w_0\right]$

$$= \left(p_\gamma(w_2) + i\ q_\gamma(w_2)\right)\cdot\left(\gamma_2 - \gamma_0\right) + \left(p_\delta(w_2) + i\ q_\delta(w_2)\right)\cdot\left(\delta_2 - \delta_0\right)$$

$$= \left[p_\gamma(w_2) + i\ q_\gamma(w_2)\right]\cdot\left[\gamma_2 - \gamma_0\right] + i\ \left(p_\gamma(w_2) + i\ q_\gamma(w_2)\right)\cdot\left(\delta_2 - \delta_0\right)$$

$$= \left[p_\gamma(w_2) + i\ q_\gamma(w_2)\right]\cdot\left[(\gamma_2 - \gamma_0) + i\ (\ \delta_2 - \delta_0)\right]$$

$$= \left[p_\gamma(w_2) + i\ q_\gamma(w_2)\right]^T\left[w_2 - w_0\right] = (g'(w_2))^T\ (w_2 - w_0).$$

Using $\zeta_2 = g'(w_2)$ we get

$$\left[\nabla_1 p(w_2) + i\,\nabla_1 q(w_2)\right]\cdot\left[w_2 - w_0\right] = \zeta_2^T (w_2 - w_0) \quad (4.1.24)$$

Similarly one can show that

$$\left[\nabla U(z_0) + i\,\nabla V(z_0)\right]\cdot\left[z_2 - z_0\right] = \left[f'(z_0)\right]^T\left[z_2 - z_0\right] \quad (4.1.25)$$

$\zeta = g'(w)$ and as the inverse of $g'(w)$ exists,

therefore $w = (g')^{-1}(\zeta) = J(\zeta) + i\,N(\zeta)$

Now $w_2 - w_0 = \left[J(\zeta_2) - J(\zeta_0)\right] + i\left[N(\zeta_2) - N(\zeta_0)\right]$

$$\begin{pmatrix} \gamma_2 - \gamma_0 \\ \delta_2 - \delta_0 \end{pmatrix} = M_1\left\{\left[\zeta_2 - \theta_3(\zeta_2 - \zeta_0)\right], \left[\zeta_2 - \theta_4(\zeta_2 - \zeta_0)\right]\right\}\begin{pmatrix} \xi_2 - \xi_0 \\ \eta_2 - \eta_0 \end{pmatrix} \quad (4.1.26)$$

where M_1 is the following matrix.

$$\begin{vmatrix}
\frac{\partial J_1}{\partial \xi^1} & \frac{\partial J_1}{\partial \xi^2} & \cdots & \frac{\partial J_1}{\partial \xi^n} & \frac{\partial J_1}{\partial \eta^1} & \frac{\partial J_1}{\partial \eta^2} & \cdots & \frac{\partial J_1}{\partial \eta^n} \\
\frac{\partial J_2}{\partial \xi^1} & \frac{\partial J_2}{\partial \xi^2} & \cdots & \frac{\partial J_2}{\partial \xi^n} & \frac{\partial J_2}{\partial \eta^1} & \frac{\partial J_2}{\partial \eta^2} & \cdots & \frac{\partial J_2}{\partial \eta^n} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\frac{\partial J_n}{\partial \xi^1} & \frac{\partial J_n}{\partial \xi^2} & \cdots & \frac{\partial J_n}{\partial \xi^n} & \frac{\partial J_n}{\partial \eta^1} & \frac{\partial J_n}{\partial \eta^2} & \cdots & \frac{\partial J_n}{\partial \eta^n} \\
\frac{\partial N_1}{\partial \xi^1} & \frac{\partial N_1}{\partial \xi^2} & \cdots & \frac{\partial N_1}{\partial \xi^n} & \frac{\partial N_1}{\partial \eta^1} & \frac{\partial N_1}{\partial \eta^2} & \cdots & \frac{\partial N_1}{\partial \eta^n} \\
\frac{\partial N_2}{\partial \xi^1} & \frac{\partial N_2}{\partial \xi^2} & \cdots & \frac{\partial N_2}{\partial \xi^n} & \frac{\partial N_2}{\partial \eta^1} & \frac{\partial N_2}{\partial \eta^2} & \cdots & \frac{\partial N_2}{\partial \eta^n} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\frac{\partial N_n}{\partial \xi^1} & \frac{\partial N_n}{\partial \xi^2} & \cdots & \frac{\partial N_n}{\partial \xi^n} & \frac{\partial N_n}{\partial \eta^1} & \frac{\partial N_n}{\partial \eta^2} & \cdots & \frac{\partial N_n}{\partial \eta^n}
\end{vmatrix}$$

J and N are evaluated at $[\zeta_2-\theta_3(\zeta_2-\zeta_0)]$ and $[\zeta_2-\theta_4(\zeta_2-\zeta_0)]$ respectively.

Substituting the values from (4.1.24) (4.1.25), (4.1.26) and using (4.1.21), (4.1.22), in (4.1.23)

$$F(Z_2,W_2,\zeta_2)-F(Z_0,W_0,\zeta_0)=k\ \mathrm{Re}\Big[-\Big(\big\{\nabla_1 p\,[W_2-\theta_1 k(W_1-W_0)]-\nabla_1 p(W_2)\big\}$$

$$+\,i\big\{\nabla_1 q\,[W_2-\theta_1 k(W_1-W_0)]-\nabla_1 q(W_2)\big\}\Big)M\big([\zeta_2-\theta_3 k(\zeta_1-\zeta_0)]\,[\zeta_2-\theta_4 k(\zeta_1-\zeta_0)]\big)$$

$$\begin{pmatrix}\xi_2-\xi_0\\ \eta_2-\eta_0\end{pmatrix}+\Big(\big\{\nabla U\,[Z_0+\theta_1 k(Z_1-Z_0)]-\nabla U(Z_0)\big\}+i\big\{\nabla V\,[Z_0+\theta_1 k(Z_1-Z_0)]-\nabla V(Z_0)\big\}\Big)$$

$$(Z_1-Z_0)+\Big\{(\zeta_1-\zeta_0)^T W_0+[f'(Z_0)]^T\,[Z_1-Z_0]\Big\}\Big] \qquad (4.1.27)$$

The last two terms of (4.1.27) are independent of k, $\nabla_1 p$, $\nabla_1 q$ and ∇U, ∇V are continuous, taking k arbitrarily small and using (4.1.20) the right hand side of (4.1.27) can be made negative.

Hence $F(Z_2,W_2,\zeta_2)-F(Z_0,W_0,\zeta_0)<0$.

This is contradictory to our hypothesis that (Z_0,W_0,ζ_0) minimizes the problem B therefore (Z_0,ζ_0) is minimizing the problem L_2.

Theorem (4.1B) (Duality Theorem)

(i) If (Z_o, W_o) is an optimal solution for B then (Z_o, W_o) is also an optimal solution for $\tilde{\tilde{B}}$ and $F(Z_o, W_o) = G(Z_o, W_o)$

(ii) If (λ_o, μ_o) is an optimal solution for $\tilde{\tilde{B}}$ then (λ_o, μ_o) is also an optimal solution for B and $G(\lambda_o, \mu_o) = F(\lambda_o, \mu_o)$.

Proof. It follows from lemma I that if (Z_o, W_o, γ_o) is a minimising solution to problem B then (Z_o, γ_o) minimizes the linear programming problem L_2. The dual of the problem L_2, according to $\tilde{\tilde{L}}$ given in Chapter III is

D.P.$(\tilde{\tilde{L}}_2)$

$$\text{maximise} \quad K(\mu) = \text{Re}\,(b^* \mu), \tag{4.1.28}$$

$$\text{subject to} \quad \left|\text{Arg}\,(-A^* \mu + \overline{[f'(Z_o)]})\right| \leq \frac{\pi}{2} - \alpha \tag{4.1.29}$$

$$\left|\text{Arg}\,\mu\right| \leq \frac{\pi}{2} - \beta \tag{4.1.30}$$

$$\mu = W_o \tag{4.1.31}$$

Clearly $\mu = W_o$ is a maximising solution to this problem. Using the duality theory of linear programming in complex space [59]

$$\text{Re}\,(b^* W_o) = \text{Re}\left\{\gamma_o^T W_o + [f'(Z_o)]^T Z_o\right\} . \tag{4.1.32}$$

From constraints (4.1.29) and (4.1.30) $\lambda = Z_o$ and $\mu = W_o$ is feasible for problem $\tilde{\tilde{B}}$. Let λ, μ be any other feasible solution of $\tilde{\tilde{B}}$. Then

$$G(Z_o,W_o) - G(\lambda,\mu) = \text{Re}\Big[-g(W_o) + f(Z_o) - [f'(Z_o)]^T Z_o + b^* W_o$$

$$+ g(\mu) - f(\lambda) + [f'(\lambda)]^T \lambda - b^* \mu\Big]$$

$$= \text{Re}\Big[-g(W_o) + g(\mu) + f(Z_o) - f(\lambda) + [f'(\lambda)]^T \lambda - b^* \mu$$

$$[g'(W_o)]^T W_o\Big] \quad \text{(using 4.1.32)} \qquad (4.1.33)$$

For feasible (Z_o,W_o) and μ we get from (4.1.2) and (4.1.6)

$$\text{Re}\left[\mu, \left(AZ_o + \overline{[g'(W_o)]} - b\right)\right] \geqslant 0. \qquad (4.1.34)$$

Similarly for feasible λ, μ and Z_o we have from (4.1.5) and (4.1.3)

$$\text{Re}\left[\left(-A^*\mu + \overline{[f'(\lambda)]}\right), Z_o\right] \geqslant 0 \qquad (4.1.35)$$

Using $(\mu, AZ_o) = (A^*\mu, Z_o)$, in (4.1.34) and (4.1.35)

$$\text{Re}\left\{\left([f'(\lambda)]^T Z_o\right) + \left([g'(W_o)]^T \mu\right) - b^*\mu\right\} \geqslant 0. \qquad (4.1.36)$$

Using (4.1.36) in (4.1.33)

$$G(Z_o,W_o) - G(\lambda,\mu) \geqslant \text{Re}\Big\{-g(W_o) + g(\mu) + f(Z_o) - f(\lambda)$$

$$+ [f'(\lambda)]^T\lambda + [g'(W_o)]^T W_o - [f'(\lambda)]^T Z_o - [g'(W_o)]^T \mu\Big\}$$

$$\geqslant \text{Re}\Big\{g(\mu) - g(W_o) - [g'(W_o)]^T[\mu - W_o] + f(Z_o) - f(\lambda) - [f'(\lambda)]^T [Z_o - \lambda]\Big\}$$

$\geqslant \quad 0$ by convexity of $\text{Re}[g(\mu)]$ and $\text{Re}[f(Z)]$.

This shows that (Z_o,W_o) is also a maximizing solution for the problem $\tilde{D}$. Now

$$G(Z_o, W_o) = \text{Re}\left[-g(W_o) + f(Z_o) - [f'(Z_o)]^T \; Z_o \; + b^* W_o\right]$$

$$= \text{Re}\left[-g(W_o) + f(Z_o) + [g'(W_o)]^T \; W_o\right] \quad \text{using (4.1.32)}$$

$$= F(Z_o, W_o),$$

which verify the equality of objective function.

The proof of the second part of the theorem follows from the symmetry of the problems.

Theorem (4.10) (Optimality Condition). If the respective problem possess feasible $(\tilde{Z}, \tilde{W})$ and $(\tilde{\lambda}, \tilde{\mu})$ for which

$$\text{Re}\left[[g'(\tilde{W})]^T \tilde{W} - g(\tilde{W}) + f(\tilde{Z})\right] = \text{Re}\left[-g(\tilde{\mu}) + f(\tilde{\lambda}) - [f'(\tilde{\lambda})]^T \tilde{\lambda} \; + b^* \tilde{\mu}\right]$$

then $(\tilde{Z}, \tilde{W})$ and $(\tilde{\lambda}, \tilde{\mu})$ are optimal feasible solutions.

Proof. From weak duality theorem we have for any feasible solution (Z, W) for the problem B

$$\text{Re}\left[[g'(W)]^T W - g(W) + f(Z)\right] \geq \text{Re}\left[-g(\tilde{\mu}) + f(\tilde{\lambda}) - [f'(\tilde{\lambda})]^T \tilde{\lambda} \; + b^* \tilde{\mu}\right]$$

$$= \text{Re}\left[[g'(\tilde{W})]^T \tilde{W} - g(\tilde{W}) + f(\tilde{Z})\right]$$

or $$\text{Re}\left\{[g'(W)]^T W - g(W) + f(Z)\right\} \geq \text{Re}\left\{[g'(\tilde{W})]^T \tilde{W} - g(\tilde{W}) + f(\tilde{Z})\right\}$$

This shows that $(\tilde{Z}, \tilde{W})$ is an optimal solution for problem B.

Similarly from weak duality theorem for any feasible solution (λ, μ) for the problem $\tilde{B}$

$$\mathrm{Re}\left[-g(\mu)+f(\lambda)-[f'(\lambda)]^T\lambda \quad +b^*\mu\right] \leq \mathrm{Re}\left[[g'(\tilde{w})]^T\,\tilde{w} - g(\tilde{w}) + f(\tilde{z})\right]$$

$$= \mathrm{Re}\left\{-g(\tilde{\mu})+f(\tilde{\lambda})-[f'(\tilde{\lambda})]^T\tilde{\lambda} \quad + b^*\tilde{\mu}\right\}$$

or $\quad \mathrm{Re}\left[-g(\mu)+f(\lambda)-[f'(\lambda)]^T\lambda \quad + b^*\mu\right] \leq \mathrm{Re}\left[-g(\tilde{\mu})+f(\tilde{\lambda})-\right.$

$$\left.[f'(\tilde{\lambda})]^T\,\tilde{\lambda} \quad + b^*\tilde{\mu}\right]$$

This shows that $(\tilde{\lambda},\tilde{\mu})$ is an optimal solution for the problem $\tilde{\tilde{B}}$.

<u>Theorem (4.1D) (Complementary Slackness)</u> (Z_o,W_o) is a joint solution iff (i) and (ii) hold

(i) $\quad \mathrm{Re}\left[W_o,\ (AZ_o + \overline{[g'(W_o)]} - b)\right] \quad = 0$

(ii) $\quad \mathrm{Re}\left[(-A^*W_o + \overline{[f'(Z_o)]} \quad),\ Z_o\right] \quad = 0.$

<u>Proof.</u> If (Z_o,W_o) is a joint solution then from the constraints we have

$$\mathrm{Re}\left[W_o,\ (AZ_o + \overline{[g'(W_o)]} - b\right] \geqslant 0, \tag{4.1.37}$$

and

$$\mathrm{Re}\left[(-A^*W_o + \overline{[f'(Z_o)]} \quad),\ Z_o\right] \geqslant 0. \tag{4.1.38}$$

Now for a joint solution the equality of objective functions give

$$\mathrm{Re}\left\{-g(W_o)+f(Z_o)+[g'(W_o)]^T \quad W_o\right\}$$

$$= \mathrm{Re}\left\{-g(W_o)+f(Z_o) - [f'(Z_o)]^T \quad Z_o \quad + b^*W_o\right\}$$

or $\quad \mathrm{Re}\left\{[g'(W_o)]^T \quad W_o + [f'(Z_o)]^T \quad Z_o \quad - b^*W_o\right\} = 0 \qquad (4.1.39)$

Adding (4.1.37) (4.1.38) and using $(W_o, AZ_o) = (A^*W_o, Z_o)$ and (4.1.39)

$$\operatorname{Re}\left\{[g'(W_o)]^T W_o + [f'(Z_o)]^T Z_o - b^*W_o\right\} = 0 .$$

Since each of (4.1.37) and (4.1.38) is nonnegative therefore we get

$$\operatorname{Re}\left[W_o, \left(AZ_o + \overline{[g'(W_o)]} - b\right)\right] = 0$$

and $$\operatorname{Re}\left[\left(-A^*W_o + \overline{[f'(Z_o)]}\right), Z_o\right] = 0.$$

Conversely if $\operatorname{Re}\left[W_o, \left(AZ_o + \overline{[g'(W_o)]} - b\right)\right] = 0$

and

$$\operatorname{Re}\left[\left(-A^*W_o + \overline{[f'(Z_o)]}\right), Z_o\right] = 0$$

We get by adding these, and using $(W_o, AZ_o) = (A^*W_o, Z_o)$

$$\operatorname{Re}\left[[g'(W_o)]^T W_o + [f'(Z_o)]^T Z_o - b^*W_o\right] = 0,$$

or $$\operatorname{Re}\left[-g(W_o) + f(Z_o) + [g'(W_o)]^T W_o\right]$$

$$= \operatorname{Re}\left[-g(W_o) + f(Z_o) - [f'(Z_o)]^T Z_o + b^*W_o\right] ,$$

which shows that (Z_o, W_o) is a joint solution.

SECTION - II

Symmetric Dual and Self Dual Programs in Complex Space

Consider the following pair of programs.

Primal Program (B)

$$\text{minimize} \quad F(Z,W) = \operatorname{Re}\left\{K(Z,W) - \left[K_2(Z,W)\right]^T W\right\} \tag{4.2.1}$$

$$\text{subject to} \quad \left|\operatorname{Arg}\ \overline{\left(-K_2(Z,W)\right)}\right| \leq \beta \tag{4.2.2}$$

$$\left|\operatorname{Arg} Z\right| \leq \alpha \tag{4.2.3}$$

Dual Program $(\tilde{B})$

$$\text{maximize} \quad G(Z,W) = \operatorname{Re}\left\{K(Z,W) - \left[K_1(Z,W)\right]^T Z\right\} \tag{4.2.4}$$

$$\text{subject to} \quad \left|\operatorname{Arg}\ \overline{\left(K_1(Z,W)\right)}\right| \leq \frac{\pi}{2} - \alpha \tag{4.2.5}$$

$$\left|\operatorname{Arg} W\right| \leq \frac{\pi}{2} - \beta \tag{4.2.6}$$

$K(Z,W)$ is a complex valued differentiable function of Z and W, Z and W are n and m dimensional complex vectors. $K_1(Z,W)$ is the gradient vector of $K(Z,W)$ with respect to Z at (Z,W) point and $K_2(Z,W)$ is the gradient vector of $K(Z,W)$ with respect to W at the point (Z,W). T denotes the transpose and $\overline{K(Z,W)}$ denotes the conjugate of $K(Z,W)$. $K(Z,W) = U(X,Y;\xi,\eta) + iV(X,Y;\xi,\eta) = U(Z,W) + i\,V(Z,W)$.

For proving the duality theorem between the primal program (B) and dual program $(\tilde{B})$ the following assumptions are needed about $K(Z,W)$.

I $\quad K(Z,W)$ has continuous first partial derivatives

II $\quad$ For each fixed W in the region defined by primal and dual constraints $\operatorname{Re}\left[K(Z,W)\right] = U(X,Y;\xi,\eta)$ is convex in X and Y such that $X + iY = Z \in C_I$ and $\xi + i\eta = W \in C_{II}$.

III $\quad$ For each fixed Z in the region defined by primal and dual constraints $\operatorname{Re}\left[K(Z,W)\right] = U(X,Y;\xi,\eta)$ is concave in ξ and η such that $\xi + i\eta = W \in C_{II}$ and $X + iY = Z \in C_I$.

IV $\quad Z^I$ be the set of all Z for which a vector W exists such that constraints (4.2.5) and (4.2.6) hold. W^I be the set of all W for which a Z exists such that (4.2.5) and (4.2.6) hold.

Define $\quad \theta(Z,W) = K_1(Z,W)$ $\quad$ (4.2.7)

Let $\mathbb{H}$ be the set of all vectors $\theta(Z,W)$ with $Z \in Z^I$ $W \in W^I$. It will be assumed that there exists a differentiable function that determines for a given $\theta_1 \in \mathbb{H}$ and $W_1 \in W^I$ a unique $Z_1 \in Z^I$ such that $\theta_1 = K_1(Z_1,W_1)$.

V $\quad W^2$ be the set of all W for which these exists a Z such that constraints (4.2.2) and (4.2.3) hold. Let Z^2 be the set of all Z for which there exists a W such that (4.2.2) and (4.2.3) hold

Define $\quad \varphi(Z,W) = K_2(Z,W)$ $\quad$ (4.2.8)

Let φ^I be the set of all $\varphi(Z,W)$ with $Z \in Z^2$, $W \in W^2$. It will be assumed that there exists a differentiable function that determines for a given $\varphi_2 \in \varphi^I$ and $Z_2 \in Z^2$ a unique $W_2 \in W^2$ such that $\varphi_2 = K_2(Z_2, W_2)$.

<u>**Lemma I.**</u> If $\mathrm{Re}\left[K(Z,W)\right] = U(X,Y;\xi,\eta)$ is convex function of X and Y such that $X + iY = Z \in C_I$ then for any two vectors Z and $\tilde{Z}$.

$$\mathrm{Re}\left[K(Z,W) - K(\tilde{Z},W)\right] \geqslant \mathrm{Re}\left\{\left[K_1(\tilde{Z},W)\right]^T \left[Z - \tilde{Z}\right]\right\} .$$

<u>**Proof**</u> Since $U(X,Y;\xi,\eta)$ is convex function of X and Y we have

$$U(X,Y;\xi,\eta) - U(\tilde{X},\tilde{Y};\xi,\eta) \geqslant U_X(\tilde{X},\tilde{Y};\xi,\eta)^T\left[X - \tilde{X}\right] +$$

$$U_Y(\tilde{X},\tilde{Y};\xi,\eta)^T\left[Y - \tilde{Y}\right]$$

$$= \left[U_X(\tilde{X},\tilde{Y};\xi,\eta)\right]^T\left[X - \tilde{X}\right] - \left[V_X(\tilde{X},\tilde{Y};\xi,\eta)\right]^T\left[Y - \tilde{Y}\right]$$

(using Cauchy Riemann conditions)

$$= \mathrm{Re}\left[U_X(\tilde{X},\tilde{Y};\xi,\eta) + i\,V_X(\tilde{X},\tilde{Y};\xi,\eta)\right]^T\left[(X - \tilde{X}) + i\,(Y - \tilde{Y})\right]$$

$$= \mathrm{Re}\left\{\left[K_1(\tilde{Z},W)\right]^T\left[Z - \tilde{Z}\right]\right\}$$

or

$$\mathrm{Re}\left\{K(Z,W) - K(\tilde{Z},W)\right\} \geqslant \mathrm{Re}\left\{\left[K_1(\tilde{Z},W)\right]^T\left[Z - \tilde{Z}\right]\right\}.$$

Hence proved.

Similarly one can show that if $\mathrm{Re}\left[K(Z,W)\right] = U(X,Y;\xi,\eta)$ is concave function of ξ and η such that $\xi + i\eta = W \in C_{II}$ then for any two vectors W and $\tilde{W}$

$$\mathrm{Re}\left[K(Z,W) - K(Z,\tilde{W})\right] \leqslant \mathrm{Re}\left\{\left[K_2(Z,\tilde{W})\right]^T\left[W - \tilde{W}\right]\right\} .$$

<u>**(Theorem 4.2A) (Weak Duality Theorem)**</u> If K is a differentiable and $\mathrm{Re}\left[K(Z,W)\right]$ is convex concave function then

$$\sup_{\tilde{D}} G(\tilde{Z},\tilde{W}) \leqslant \inf_{D} F(Z,W)$$

Proof. Since K is differentiable and Re $\left[K(Z,W)\right]$ is convex concave therefore we have

$$\operatorname{Re}\left[K(Z,\tilde{W}) - K(\tilde{Z},\tilde{W})\right] \geqslant \operatorname{Re}\left\{\left[K_1(\tilde{Z},\tilde{W})\right]^T \left[Z - \tilde{Z}\right]\right\} \tag{4.2.9}$$

$$\operatorname{Re}\left[K(Z,\tilde{W}) - K(Z,W)\right] \leqslant \operatorname{Re}\left\{\left[K_2(Z,W)\right]^T \left[\tilde{W} - W\right]\right\} \tag{4.2.10}$$

From (4.2.9) and (4.2.10)

$$\operatorname{Re}\left[K(Z,W) - K(\tilde{Z},\tilde{W})\right] \geqslant \operatorname{Re}\left\{\left[K_1(\tilde{Z},\tilde{W})\right]^T \left[Z - \tilde{Z}\right] - \left[K_2(Z,W)\right]^T \left[\tilde{W} - W\right]\right\}$$

$$\operatorname{Re}\left\{\left[K(Z,W) - \left[K_2(Z,W)\right]^T W\right] - \left[K(\tilde{Z},\tilde{W}) - (K_1(\tilde{Z},\tilde{W}))^T Z\right]\right\}$$

$$\geqslant \operatorname{Re}\left\{(K_1(\tilde{Z},\tilde{W}))^T Z - (K_2(Z,W))^T \tilde{W}\right\} \tag{4.2.11}$$

Let (Z,W) is feasible for primal problem (B) and $(\tilde{Z},\tilde{W})$ is feasible for dual problem ($\tilde{B}$) then from constraints (4.2.2) and (4.2.6)

$$\operatorname{Re}\left\{-\left[K_2(Z,W)\right]^T \tilde{W}\right\} \geqslant 0 \tag{4.2.12}$$

Similarly from (4.2.3) and (4.2.5)

$$\operatorname{Re}\left\{\left[K_1(\tilde{Z},\tilde{W})\right]^T Z\right\} \geqslant 0 \tag{4.2.13}$$

From (4.2.12) and (4.2.13)

$$\operatorname{Re}\left\{\left[K_1(\tilde{Z},\tilde{W})\right]^T Z - \left[K_2(Z,W)\right]^T \tilde{W}\right\} \geqslant 0. \tag{4.2.14}$$

Using (4.2.14) in (4.2.11)

$$\operatorname{Re}\left\{K(Z,W) - \left[K_2(Z,W)\right]^T W\right\} - \operatorname{Re}\left\{K(\tilde{Z},\tilde{W}) - \left[K_1(\tilde{Z},\tilde{W})\right]^T \tilde{Z}\right\} \geqslant 0$$

$$F(Z,W) \geqslant G(\tilde{Z},\tilde{W})$$

$$\text{Inf } F(Z,W) \geqslant \text{Sup } G(\tilde{Z},\tilde{W}).$$

<u>Theorem (4.2B)</u> (<u>Optimality Condition</u>) If the respective problem possess feasible $(\tilde{Z},\tilde{W})$ and (Z,W) for which

$$\text{Re}\left\{K(\tilde{Z},\tilde{W}) - K_2(\tilde{Z},\tilde{W})^T\,\tilde{W} = \text{Re}\left\{K(Z,W) - [K_1(Z,W)]^T Z\right\}\right. \tag{4.2.15}$$

then $(\tilde{Z},\tilde{W})$ and (Z,W) are optimal feasible solutions.

<u>Proof</u>. From weak duality theorem we have that for any feasible solution $(\tilde{\tilde{Z}},\tilde{\tilde{W}})$ for the primal problem (B)

$$\text{Re}\left\{K(\tilde{\tilde{Z}},\tilde{\tilde{W}}) - [K_2(\tilde{\tilde{Z}},\tilde{\tilde{W}})]^T\,\tilde{\tilde{W}}\right\} \geqslant \text{Re}\left\{K(Z,W) - [K_1(Z,W)]^T Z\right\}$$

$$\text{Re}\left\{K(\tilde{Z},\tilde{W}) - [K_2(\tilde{Z},\tilde{W})]^T\,\tilde{W}\right\}$$

or $$\text{Re}\left\{K(\tilde{\tilde{Z}},\tilde{\tilde{W}}) - [K_2(\tilde{\tilde{Z}},\tilde{\tilde{W}})]^T\,\tilde{\tilde{W}}\right\} \geqslant \text{Re}\left\{K(\tilde{Z},\tilde{W}) - [K_2(\tilde{Z},\tilde{W})]^T\,\tilde{W}\right\}$$

It shows that $(\tilde{Z},\tilde{W})$ is an optimal solution for the primal problem. Similarly it can be shown that (Z,W) is an optimal solution for the dual problem.

<u>Lemma 2</u>. If (Z_0, W_0, ξ_0) is a minimizing solution of primal problem(B) which on rephasing can be read as

<u>Primal Program</u> (B)

$$\text{minimize} \quad F(Z,W,\xi) = \text{Re}\left[K(Z,W) - \xi^T W\right] \tag{4.2.16}$$

$$\text{subject to } \left|\text{Arg}(-\bar{\xi})\right| \leq \beta \tag{4.2.17}$$

$$\left|\text{Arg } Z\right| \leq \alpha \tag{4.2.18}$$

$$\xi = K_2(Z,W) \tag{4.2.19}$$

then (Z_0, ς_0) minimizes the linear programming problem

P.P (L_2)

$$\text{minimize} \quad H(Z, \varsigma) = \text{Re}\left\{K(Z_0, W_0) - \varsigma^T W_0 + \left[K_1(Z_0, W_0)\right]^T Z\right\} \tag{4.2.20}$$

$$\text{subject to} \quad \left|\text{Arg}\,(-\bar{\varsigma})\right| \leq \beta \tag{4.2.21}$$

$$\left|\text{Arg}\, Z\right| \leq \alpha \tag{4.2.22}$$

<u>Proof</u>. (Z_0, W_0, ς_0) is a minimizing solution of primal problem (B) therefore $F(Z_0, W_0, \varsigma_0) \leq F(Z, W, \varsigma)$ for all feasible Z, W, ς satisfying the constraints of primal problem (B). The constraints (4.2.17) (4.2.18) and (4.2.21) (4.2.22) are same therefore (Z_0, ς_0) is also feasible solution to primal problem (L_2). Now suppose (Z_1, ς_1) be another feasible solution to primal problem (L_2) such that

$$H(Z_1, \varsigma_1) < H(Z_0, \varsigma_0)$$

$$\text{or} \quad \text{Re}\left[-(\varsigma_1 - \varsigma_0)^T W_0 + \left[K_1(Z_0, W_0)\right]^T \left[Z_1 - Z_0\right]\right] < 0 \tag{4.2.23}$$

$$\text{Define } Z_2 = Z_0 + \lambda(Z_1 - Z_0) \qquad \text{for } 0 < \lambda \leq 1 \tag{4.2.24}$$

$$\varsigma_2 = \varsigma_0 + \lambda(\varsigma_1 - \varsigma_0) \tag{4.2.25}$$

From lemma 2 [42] Z_2, ς_2 is also feasible for problems B and L_2.

Since $\varsigma_2 \in \varsigma^I$ there exists a $W_2 \in W^2$ such that

$$\varsigma_2 = K_2(Z_2, W_2) \qquad \text{(By assumption V)} \tag{4.2.26}$$

and thus (Z_2, W_2, ς_2) is a feasible solution of primal problem (B)

consider

$$F(z_2, w_2, \zeta_2) - F(z_0, w_0, \zeta_0) = \mathrm{Re}\Big[K(z_2, w_2) - \zeta_2^T w_2 -$$

$$K(z_0, w_0) + \zeta_0^T w_0 \Big]$$

$$= \mathrm{Re}\Big(\Big[\Big\{ \nabla_1 U\big([z_2 - \lambda_1 (z_2 - z_0)] , (w_2 - \lambda_1 (w_2 - w_0))\big) +$$

$$i \nabla_1 V\big([z_2 - \lambda_2 (z_2 - z_0)], [w_2 - \lambda_2 (w_2 - w_0)] \big)\Big\} \cdot \{z_2 - z_0\}\Big]$$

$$+ \Big[\Big\{ \nabla_2 U\big([z_2 - \lambda_3 (z_2 - z_0)], [w_2 - \lambda_3 (w_2 - w_0)]\big) +$$

$$i \nabla_2 V\big([z_2 - \lambda_4 (z_2 - z_0)], [w_2 - \lambda_4 (w_2 - w_0)]\big)\Big\} \cdot \{w_2 - w_0\}$$

$$- \zeta_2^T w_2 + \zeta_0^T w_0$$ (From 2n and 2m dimensional meanvalue theorem for

U and V, $\nabla_1 = \begin{pmatrix} D_x \\ D_y \end{pmatrix}$, $\nabla_2 = \begin{pmatrix} D_\xi \\ D_\eta \end{pmatrix}$)

$$= \mathrm{Re}\Big(\Big[\Big\{\nabla_1 U\big([z_0 + \lambda_3 (z_2 - z_0)]\ [w_0 + \lambda_3 (w_2 - w_0)]\big) + i \nabla_1 V\big([z_0 + \lambda_4 (z_2 - z_0)]$$

$$[w_0 + \lambda_4 (w_2 - w_0)]\big)\Big\} \cdot \{z_2 - z_0\}\Big]$$

$$+ \Big[\Big\{ \nabla_2 U\big([z_2 - \lambda_3 (z_2 - z_0)], [w_2 - \lambda_3 (w_2 - w_0)]\big) + i \nabla_2 V\big([z_2 - \lambda_4 (z_2 - z_0)]$$

$$[w_2 - \lambda_4 (w_2 - w_0)]\big)\Big\} \cdot \{w_2 - w_0\}\Big]$$

$$- \zeta_2^T w_2 + \zeta_0^T w_0 \Big)$$ (Taking $\lambda_1 = 1 - \lambda_3$, $\lambda_2 = 1 - \lambda_4$

(4.2.27)

Now

$$\left[\nabla_1 U(Z_o,W_o) + i\,\nabla_1 V(Z_o,W_o)\right]\cdot\left[Z_2-Z_o\right]$$

$$= \left[D_X U(Z_o,W_o)\right]^T\left[X_2-X_o\right] + i\left[D_X V(Z_o,W_o)\right]^T\left[(X_2-X_o)\right] +$$

$$\left[D_Y U(Z_o,W_o)\right]^T\left[Y_2-Y_o\right] + i\left[D_Y V(Z_o,W_o)\right]^T\left[Y_2-Y_o\right]$$

$$= \left[D_X U(Z_o,W_o) + iD_X V(Z_o,W_o)\right]^T\left[X_2-X_o\right] + i\left[D_X U(Z_o,W_o) +\right.$$

$$\left. i\,D_X V(Z_o,W_o)\right]^T\left[Y_2-Y_o\right]$$

(using Cauchy Riemann conditions)

$$= \left[D_X U(Z_o,W_o) + iD_X V(Z_o,W_o)\right]^T\left[(X_2-X_o) + i\,(Y_2-Y_o)\right]$$

$$= \left[D_X U(Z_o,W_o) + iD_X V(Z_o,W_o)\right]^T\left[Z_2-Z_o\right]$$

$$= \left[K_1(Z_o,W_o)\right]^T\left[Z_2-Z_o\right] \quad \ldots$$

Similarly

$$\left[\nabla_2 U(Z_2,W_2) + i\nabla_2 V(Z_2,W_2)\right]\cdot\left[W_2-W_o\right]$$

$$= \left[K_2(Z_2,W_2)\right]^T\left[W_2-W_o\right] = \zeta_2^T\left[W_2-W_o\right] \quad \ldots$$

Further $\zeta = K_2(Z\ ,W)$ therefore $W = f\ (Z,\zeta) = p(Z,\zeta) + iq(Z,\zeta) = \xi + i\eta$

Now $\quad W_2-W_o = \left[p(Z_2,\zeta_2) - p(Z_o,\zeta_o)\right] + i\left[q(Z_2,\zeta_2) - q(Z_o,\zeta_o)\right]$

$$= M\,([\ ,\ ;\ ,\ ])\ (Z_2-Z_o) + N([\ ,\ ;\ ,\ ])\ (\zeta_2-\zeta_o) \quad \ldots$$

or $$\begin{pmatrix} \xi_2 - \xi_0 \\ \eta_2 - \eta_0 \end{pmatrix} = M\Big\{[z_2 - \lambda_5(z_2 - z_0)]\,[\zeta_2 - \lambda_5(\zeta_2 - \zeta_0)],$$

$$[z_2 - \lambda_6\,(z_2 - z_0)],[\zeta_2 - \lambda_6\,(\zeta_2 - \zeta_0)]\Big\}\begin{pmatrix} X_2 - X_0 \\ Y_2 - Y_0 \end{pmatrix}$$

$$+ N\Big\{\,[z_2 - \lambda_5(z_2 - z_0)]\,[\zeta_2 - \lambda_5(\zeta_2 - \zeta_0)][z_2 - \lambda_6(z_2 - z_0)\,)$$

$$(\zeta_2 - \lambda_6(\zeta_2 - \zeta_0))\Big\}\begin{pmatrix} \gamma_2 - \gamma_0 \\ \delta_2 - \delta_0 \end{pmatrix}$$

(By assumption (5) and mean value theorem).

where

$$M = \begin{vmatrix}
\frac{\partial p_1}{\partial x_1} & \frac{\partial p_1}{\partial x_2} & \cdots & \frac{\partial p_1}{\partial x_n} & \frac{\partial p_1}{\partial y_1} & \frac{\partial p_1}{\partial y_2} & \cdots & \frac{\partial p_1}{\partial y_n} \\
\frac{\partial p_2}{\partial x_1} & \frac{\partial p_2}{\partial x_2} & \cdots & \frac{\partial p_2}{\partial x_n} & \frac{\partial p_2}{\partial y_1} & \frac{\partial p_2}{\partial y_2} & \cdots & \frac{\partial p_2}{\partial y_n} \\
\vdots & & & & & & & \\
\frac{\partial p_n}{\partial x_1} & \frac{\partial p_n}{\partial x_2} & \cdots & \frac{\partial p_n}{\partial x_n} & \frac{\partial p_n}{\partial y_1} & \frac{\partial p_n}{\partial y_2} & \cdots & \frac{\partial p_n}{\partial y_n} \\
\frac{\partial q_1}{\partial x_1} & \frac{\partial q_1}{\partial x_2} & \cdots & \frac{\partial q_1}{\partial x_n} & \frac{\partial q_1}{\partial y_1} & \frac{\partial q_1}{\partial y_2} & \cdots & \frac{\partial q_1}{\partial y_n} \\
\frac{\partial q_2}{\partial x_1} & \frac{\partial q_2}{\partial x_2} & \cdots & \frac{\partial q_2}{\partial x_n} & \frac{\partial q_2}{\partial y_1} & \frac{\partial q_2}{\partial y_2} & \cdots & \frac{\partial q_2}{\partial y_n} \\
\vdots & & & & & & & \\
\frac{\partial q_n}{\partial x_1} & \frac{\partial q_n}{\partial x_2} & \cdots & \frac{\partial q_n}{\partial x_n} & \frac{\partial q_n}{\partial y_1} & \frac{\partial q_n}{\partial y_2} & \cdots & \frac{\partial q_n}{\partial y_n}
\end{vmatrix}$$

$$N = \begin{vmatrix}
\frac{\partial p_1}{\partial \gamma^1} & \frac{\partial p_1}{\partial \gamma^2} & \cdots & \frac{\partial p_1}{\partial \gamma^n} & \frac{\partial p_1}{\partial \delta^1} & \frac{\partial p_1}{\partial \delta^2} & \cdots & \frac{\partial p_1}{\partial \delta^n} \\
\frac{\partial p_2}{\partial \gamma^1} & \frac{\partial p_2}{\partial \gamma^2} & \cdots & \frac{\partial p_2}{\partial \gamma^n} & \frac{\partial p_2}{\partial \delta^1} & \frac{\partial p_2}{\partial \delta^2} & \cdots & \frac{\partial p_2}{\partial \delta^n} \\
\vdots & & & & & & & \\
\frac{\partial p_n}{\partial \gamma^1} & \frac{\partial p_n}{\partial \gamma^2} & \cdots & \frac{\partial p_n}{\partial \gamma^n} & \frac{\partial p_n}{\partial \delta^1} & \frac{\partial p_n}{\partial \delta^2} & \cdots & \frac{\partial p_n}{\partial \delta^n} \\
\frac{\partial q_1}{\partial \gamma^1} & \frac{\partial q_1}{\partial \gamma^2} & \cdots & \frac{\partial q_1}{\partial \gamma^n} & \frac{\partial q_1}{\partial \delta^1} & \frac{\partial q_1}{\partial \delta^2} & \cdots & \frac{\partial q_1}{\partial \delta^n} \\
\frac{\partial q_2}{\partial \gamma^1} & \frac{\partial q_2}{\partial \gamma^2} & \cdots & \frac{\partial q_2}{\partial \gamma^n} & \frac{\partial q_2}{\partial \delta^1} & \frac{\partial q_2}{\partial \delta^2} & \cdots & \frac{\partial q_2}{\partial \delta^n} \\
\vdots & & & & & & & \\
\frac{\partial q_n}{\partial \gamma^1} & \frac{\partial q_n}{\partial \gamma^2} & \cdots & \frac{\partial q_n}{\partial \gamma^n} & \frac{\partial q_n}{\partial \delta^1} & \frac{\partial q_n}{\partial \delta^2} & \cdots & \frac{\partial q_n}{\partial \delta^n}
\end{vmatrix}$$

p is evaluated at $\left([Z_2 - \lambda_5(Z_2 - Z_o)], [\rho_2 - \lambda_5(\rho_2 - \rho_o)]\right)$ and q is evaluated at $\left([Z_2 - \lambda_6(Z_2 - Z_o)], [\rho_2 - \lambda_6(\rho_2 - \rho_o)]\right)$.

Adding $Re\left[\nabla_1 U(Z_o,W_o) + i\,\nabla_1 V(Z_o,W_o)\right]^T \left[Z_2 - Z_o\right]$, $Re\left[\nabla_2 U(Z_2,W_2) + i\,\nabla_2 V(Z_2,W_2)\right]^T \left[W_2 - W_o\right]$

and substracting $Re\left[\nabla_1 U(Z_o,W_o) + i\,\nabla_1 V(Z_o,W_o)\right]^T \left[Z_2 - Z_o\right]$, $Re\left[\nabla_2 U(Z_2\ W_2) + i\,\nabla_2 V(Z_2,W_2)\right]^T \left[W_2 - W_o\right]$ from (4.2.27) and using (4.2.24), (4.2.25),

$$F(Z_2,W_2,\varsigma_2)-F(Z_o,W_o,\varsigma_o)=\lambda\,\mathrm{Re}\Big\{\Big(\Big[\nabla_1 U\Big([Z_o+\lambda_3\lambda\,(Z_1-Z_o)]$$

$$[W_o+\lambda_3\lambda\,(W_1-W_o)]\Big)$$

$$-\nabla_1 U(Z_o,W_o)\Big]+i\Big[\nabla_1 V\Big([Z_o+\lambda_4\lambda\,(Z_1-Z_o)]\,,[W_o+\lambda_4\lambda\,(W_1-W_o)]\Big)$$

$$-\nabla_1 V(Z_o,W_o)\Big]\Big\}\cdot\Big\{Z_1-Z_o\Big\}$$

$$+\Big[\Big(\Big\{\nabla_2 U\Big([Z_2-\lambda_3\lambda\,(Z_1-Z_o)]\,[W_2-\lambda_3\lambda\,(W_1-W_o)]\Big)-\nabla_2 U(Z_2,W_2)\Big\}+$$

$$i\Big\{\nabla_2 V\Big([Z_2-\lambda_4\lambda\,(Z_1-Z_o)]\,[W_2-\lambda_4\,(W_1-W_o)]\Big)-\nabla_2 V(Z_2,W_2)\Big\}\Big]$$

$$M\Big\{(Z_2-\lambda_5\lambda\,(Z_1-Z_o)]\,[\varsigma_2-\lambda_5\lambda\,(\varsigma_1-\varsigma_o)]\,[Z_2-\lambda_6\lambda\,(Z_1-Z_o)]$$

$$[\varsigma_2-\lambda_6\lambda\,(\varsigma_1-\varsigma_o)]\Big]\Big\}\cdot\Big\{Z_1-Z_o\Big\}$$

$$+N\Big\{(Z_2-\lambda_5\lambda\,(Z_1-Z_o)]\,[\varsigma_2-\lambda_5\lambda\,(\varsigma_1-\varsigma_o)]\,[Z_2-\lambda_6\lambda\,(Z_1-Z_o)]$$

$$[\varsigma_2-\lambda_6\lambda(\varsigma_1-\varsigma_o)]\Big\}\cdot\Big\{\varsigma_1-\varsigma_o\Big\}\Big)\Big]$$

$$-(\varsigma_1-\varsigma_o)^2 W_o+\Big[K_1\,(Z_o,W_o)\Big]^T\,\Big[Z_1-Z_o\Big]\Big)$$

$\nabla_1 U$, $\nabla_1 V$; $\nabla_2 U$, $\nabla_2 V$ are continuous therefore it follows that for a sufficiently small λ, $F(Z_2,W_2,\varsigma_2)-F(Z_o,W_o,\varsigma_o)$ will have the same sign as $\mathrm{Re}\Big\{-(\varsigma_1-\varsigma_o)^2\,W_o+\Big[K_1(Z_o,W_o)\Big]^T\,\Big[Z_1-Z_o\Big]\Big\}$ which by (4.2.23) is negative. This contradicts the hypotheses that (Z_o, W_o, ς_o) is an optimal solution to primal problem B. Hence (Z, ς_o) minimizes the linear problem (L_2).

Hence proved.

Theorem (4.2.e) (Duality Theorem) If (Z_0, W_0) is an optimal solution for primal program (B) or dual program ($\tilde{\tilde{B}}$)

(a) (Z_0, W_0) is optimal for both primal and dual problems.

(b) The minimum of primal problem (B) equals the maximum of dual problem ($\tilde{\tilde{B}}$).

Proof It has been shown in lemma 2 that if $(Z_0, W_0, \mathcal{S}_0)$ is an optimal solution to primal problem (B) then $(Z_0, \mathcal{S}_0)$ is optimal to the linear programming problem (L_2).

The dual of (L_2) is

Dual Problem ($\tilde{\tilde{L}}_2$)

maximise $\quad \mathrm{Re}\ \left[K(Z_0, W_0)\right]$

$$\text{subject to} \quad \left|\mathrm{Arg}\ \left(K_1(Z_0, W_0)\right)\right| \leq \frac{\pi}{2} - \alpha \tag{4.2.28}$$

$$\left|\mathrm{Arg}\ \mu\right| \leq \frac{\pi}{2} - \beta \tag{4.2.29}$$

$$\mu = W_0 \tag{4.2.30}$$

Now by the duality theorem of linear programming in complex space given by Levinson [59] dual problem $(\tilde{\tilde{L}}_2)$ has an optimal solution, and from (4.2.30) $\mu = W_0$ is that solution therefore

$$\mathrm{Re}\ \left[K(Z_0, W_0)\right] = \mathrm{Re}\ \left\{K(Z_0, W_0) - \mathcal{S}_0^T W_0 + \left[K_1(Z_0, W_0)\right]^T Z_0\right\}$$

$$\mathrm{Re}\left\{\left[K_1(Z_0, W_0)\right]^T Z_0 - \mathcal{S}_0^T W_0\right\} = 0$$

$$\mathrm{Re}\left\{\left[K_1(Z_0, W_0)\right]^T Z_0 - \left[K_2(Z_0, W_0)\right]^T W_0\right\} = 0 \tag{4.2.31}$$

Since (Z_o, W_o) satisfies (4.2.28) and (4.2.29) therefore it is feasible for dual problem ($\tilde{\tilde{B}}$). Let (Z_1, W_1) be any other feasible solution to dual problem ($\tilde{\tilde{B}}$).

For convexity and concavity of Re $[K(Z,W)]$ we get

$$\operatorname{Re}\left[K(Z_o,W_1) - K(Z_1,W_1)\right] \geqslant \operatorname{Re}\left\{\left[K_1(Z_1,W_1)\right]^T \left[Z_o - Z_1\right]\right\}$$

$$\operatorname{Re}\left[K(Z_o,W_o) - K(Z_o,W_1)\right] \geqslant \operatorname{Re}\left\{\left[K_2(Z_o,W_o)\right]^T \left[W_o - W_1\right]\right\}$$

or

$$\operatorname{Re}\left[K(Z_o,W_o) - K(Z_1,W_1)\right] \geqslant \operatorname{Re}\left\{\left[K_1(Z_1,W_1)\right]^T \left[Z_o - Z_1\right] + \left[K_2(Z_o,W_o)\right]^T \left[W_o - W_1\right]\right\} \tag{4.2.32}$$

Now

$$\begin{aligned} G(Z_o,W_o) - G(Z_1,W_1) &= \operatorname{Re}\Big[K(Z_o,W_o) - \left[K_1(Z_o,W_o)\right]^T Z_o - K(Z_1,W_1) + \left[K_1(Z_1,W_1)\right]^T Z_1\Big] \\ &\geqslant \operatorname{Re}\Big[\left[K_1(Z_1,W_1)\right]^T \left[Z_o - Z_1\right] + \left[K_2(Z_o,W_o)\right]^T \left[W_o - W_1\right] - \left[K_1(Z_o,W_o)\right]^T Z_o \\ &\qquad + \left[K_1(Z_1,W_1)\right]^T Z_1\Big] \quad \text{(using (4.2.32))} \\ &= \operatorname{Re}\Big[\left[K_1(Z_1,W_1)\right]^T Z_o + \left[K_2(Z_o,W_o)\right]^T \left[W_o - W_1\right] - \left[K_1(Z_o,W_o)\right]^T Z_o\Big] \\ &= \operatorname{Re}\Big[\left[K_1(Z_1,W_1)\right]^T Z_o + \left[K_2(Z_o,W_o)\right]^T \left[W_o - W_1\right] - \left[K_2(Z_o,W_o)\right]^T W_o\Big] \\ &\qquad \text{using (4.2.31)} \\ &= \operatorname{Re}\Big[\left[K_1(Z_1,W_1)\right]^T Z_o - \left[K_2(Z_o,W_o)\right]^T W_1\Big] \end{aligned} \tag{4.2.33}$$

From (4.2.2) and (4.2.6)

$$\mathrm{Re}\left\{\left[-K_2(Z_o,W_o)\right]^T W_1\right\} \geqslant 0 \tag{4.2.34}$$

From (4.2.3) and (4.2.5)

$$\mathrm{Re}\left\{\left[K_1(Z_1,W_1)\right]^T Z_o\right] \geqslant 0 \tag{4.2.35}$$

From (4.2.34) and (4.2.35)

$$\mathrm{Re}\left\{\left[K_1(Z_1,W_1)\right]^T Z_o - \left[K_2(Z_o,W_o)\right]^T W_1\right\} \geqslant 0 \tag{4.2.36}$$

Using (4.2.36) in (4.2.33) we get

$$G(Z_o,W_o) - G(Z_1,W_1) \geqslant \mathrm{Re}\left\{\left[K_1(Z_1,W_1)\right]^T Z_o - \left[K_2(Z_o,W_o)\right]^T W_1\right\} \geqslant 0$$

i.e. $$G(Z_o,W_o) - G(Z_1,W_1) \geqslant 0 \tag{4.2.37}$$

This shows that (Z_o,W_o) is an optimal solution to dual problem ($\tilde{\tilde{B}}$).

Now $$F(Z_o,W_o) = \mathrm{Re}\left[K(Z_o,W_o) - \left[K_2(Z_o,W_o)\right]^T W_o\right]$$

$$= \mathrm{Re}\left[K(Z_o,W_o) - \left[K_1(Z_o,W_o)\right]^T Z_o\right] \text{(using (4.2.31))}$$

$$= G(Z_o,W_o)$$

This shows that minimum of primal problem B equals to maximum of dual problem $\tilde{\tilde{B}}$. Converse follows from the symmetry of the program.

<u>Theorem (4.2D)</u> (<u>Complementary Slackness Theorem</u>) (Z_o,W_o) is a joint solution iff (i) and (ii) are satisfied

(i) $\quad Re\left\{\left[K_1(Z_o,W_o)\right]^T Z_o\right\} = 0$

(ii) $\quad Re\left\{\left[K_2(Z_o,W_o)\right]^T W_o\right\} = 0$

<u>Proof</u>. If (Z_o,W_o) is feasible for primal problem (B) and dual problem $(\tilde{B})$ then we have

$$Re\left[(K_1(Z_o,W_o))^T Z_o\right] \geqslant 0 \qquad (4.2.38)$$

$$Re\left[-(K_2(Z_o,W_o))^T W_o\right] \geqslant 0 \qquad (4.2.39)$$

or $$Re\left[(K_1(Z_o,W_o))^T Z_o - (K_2(Z_o,W_o))^T W_o\right] \geqslant 0 \qquad (4.2.40)$$

Since (Z_o,W_o) is a joint solution (optimal solution) we have

$$Re\left[K(Z_o,W_o) - \left[K_2(Z_o,W_o)\right]^T W_o\right] = Re\left[K(Z_o,W_o) - \left[K_1(Z_o,W_o)\right]^T Z_o\right]$$

or $$Re\left[\left[K_1(Z_o,W_o)\right]^T Z_o - \left[K_2(Z_o,W_o)\right]^T W_o\right] = 0 \qquad (4.2.41)$$

From (4.2.38) and (4.2.39) each product is nonnegative therefore from (4.2.38) (4.2.39) and (4.2.41) we get

$$Re\left\{\left[K_1(Z_o,W_o)\right]^T Z_o\right\} = 0$$

$$Re\left\{\left[K_2(Z_o,W_o)\right]^T W_o\right\} = 0$$

Conversely if $\quad Re\left\{\left[K_1(Z_o,W_o)\right]^T Z_o\right\} = 0$

and $\quad Re\left[\left\{K_2(Z_o,W_o)\right\}^T W_o\right] = 0$

Now $$F(Z_o,W_o) = \text{Re}\left\{K(Z_o,W_o) - [K_2(Z_o,W_o)]^T W_o\right\}$$

$$= \text{Re}\left[K(Z_o,W_o)\right] = \text{Re}\left[K(Z_o,W_o) - [K_1(Z_o,W_o)]^T Z_o\right]$$

$$= G\ K(Z_o,W_o)$$

It shows that (Z_o,W_o) is a joint solution.

Proof is complete.

Self Dual Nonlinear Program in Complex Space.

Consider the following pair of symmetric dual programs in complex space.

Primal Program (B)

$$\text{minimize} \quad F(Z,W) = \text{Re}\left\{K(Z,W) - [K_2(Z,W)]^T W\right\} \tag{4.2.42}$$

$$\text{subject to} \quad \left|\text{Arg}\ \left(\overline{-K_2(Z,W)}\right)\right| \leq \frac{\pi}{2} - \alpha \tag{4.2.43}$$

$$\left|\text{Arg}\ Z\right| \leq \alpha \tag{4.2.44}$$

$$\left|\text{Arg}\ W\right| \leq \alpha \tag{4.2.45}$$

Dual Program ($\tilde{\tilde{B}}$)

$$\text{maximize} \quad G(Z,W) = \text{Re}\left\{K(Z,W) - [K_1(Z,W)]^T Z\right\} \tag{4.2.46}$$

$$\text{subject to} \quad \left|\text{Arg}\ \left(\overline{K_1(Z,W)}\right)\right| \leq \frac{\pi}{2} - \alpha \tag{4.2.47}$$

$$\left|\text{Arg}\ Z\right| \leq \alpha \tag{4.2.48}$$

$$\left|\text{Arg}\ W\right| \leq \alpha \tag{4.2.49}$$

$K(Z,W)$ is differentiable skew symmetric function and α is a real vector with $0 \leq \alpha \leq \pi/2$.

<u>Self Duality Theorem.</u> Suppose $(K(Z,W)$ is differentiable and skew symmetric then primal program (B) and dual program $(\tilde{\tilde{B}})$ are formally identical if (B) and $(\tilde{\tilde{B}})$ are dual programs and if (Z_o,W_o) is a joint optimal solution then so is (W_o,Z_o) and $F(Z_o,W_o) = \mathrm{Re}\left[K(Z_o,W_o)\right] = 0$.

<u>Proof.</u> The primal program (B) can be written as

$$\text{maximize} \qquad -F(Z,W) = \mathrm{Re}\left[-K(Z,W) + \left[K_2(Z,W)\right]^T W\right]$$

$$\text{subject to} \quad \left|\mathrm{Arg}\ \left(\overline{K_1(W,Z)}\right)\right| \leq \frac{\pi}{2} - \alpha$$

$$|\mathrm{Arg}\ Z| \leq \alpha$$

$$|\mathrm{Arg}\ W| \leq \alpha$$

Since $K(Z,W)$ is differentiable skew symmetric function therefore

$$K_1(Z,W) = -K_2(W,Z)$$

The above primal program can be written as

$$\text{maximize} \quad -F(Z,W) = \mathrm{Re}\left[K(W,Z) - \left[K_1(W,Z)\right]^T W\right]$$

$$\text{subject to} \quad \left|\mathrm{Arg}\ \left(\overline{K_1(W,Z)}\right)\right| \leq \frac{\pi}{2} - \alpha$$

$$|\mathrm{Arg}\ Z| \leq \alpha$$

$$|\mathrm{Arg}\ W| \leq \alpha$$

This is just dual program ($\tilde{\tilde{B}}$). Thus (W,Z) is optimal for $\tilde{\tilde{B}}$ then (Z,W) is optimal for primal program (B). Similarly (Z,W) is optimal for primal program B then (W,Z) is optimal for dual program $\tilde{\tilde{B}}$.

If B and $\tilde{\tilde{B}}$ are dual programs and (Z W) is joint optimal solution then

$$Re\left\{\left[-K_2(Z,W)\right]^T W\right\} = Re\left\{\left[-K_1(Z,W)\right]^T Z\right\} \tag{4.2.50}$$

From (4.2.43) and (4.2.49)

$$Re\left\{\left[-K_2(Z,W)\right]^T W\right\} \geq 0 \tag{4.2.51}$$

From (4.2.44) and (4.2.47)

$$Re\left\{\left[K_1(Z,W)\right]^T Z\right\} \geq 0 \tag{4.2.52}$$

Now we have from (4.2.50) (4.2.51) and (4.2.52)

$$0 \leq Re\left\{\left[-K_2(Z,W)\right]^T W\right\} = Re\left\{\left[-K_1(Z,W)\right]^T Z\right\} \leq 0$$

Thus $Re\left\{\left[-K_2(Z,W)\right]^T W\right\} = 0$

Similarly (W,Z) is also joint optimal solution we can show

$$Re\left\{\left[-K_2(W,Z)\right]^T Z\right\} = 0$$

Hence $F(Z,W) = F(W,Z) = Re\left[K(Z,W)\right] = Re\left[K(W,Z)\right] = Re\left[-K(Z,W)\right]$

and therefore

$$F(Z,W) = Re\left[K(Z,W)\right] = 0$$

Hence proved.

Composite Program in Complex Space

minimize $\quad H(Z,W) = \mathrm{Re}\left[\left[K_1(Z,W)\right]^T Z - \left[K_2(Z,W)\right]^T W \right]$

subject to $\quad \left| \mathrm{Arg}\ \overline{(-K_2(Z,W))} \right| \leq \beta$

$$\left| \mathrm{Arg}\ \overline{(K_1(Z,W))} \right| \leq \frac{\pi}{2} - \alpha$$

$$\left| \mathrm{Arg}\ Z \right| \leq \alpha$$

$$\left| \mathrm{Arg}\ W \right| \leq \frac{\pi}{2} - \beta$$

From the constraints $\mathrm{Re}\left[\left[K_1(Z,W)\right]^T Z - \left[K_2(Z,W)\right]^T W \right] \geq 0.$

Hence any feasible vector which makes the objective function vanish must be optimal.

CHAPTER - V

Duality Theorem for Continuous Nonlinear Programming Problems*

The purpose of this chapter is to present the duality theorem for a class of continuous nonlinear programming problems. Duality theorem is proved by using the duality theorem for continuous linear programming problem [85] , [86] and adopting the technique of Dorn [21] , [25] . This chapter is divided into two sections. Section I deals with the duality theorem for continuous quadratic programming

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* The two papers on which this chapter is based were communicated to OPSEARCH early in 1968. One was even accepted for publication. Later the papers concerned with the same subject by Hansen [43,44] came to the notice of the author and accordingly the papers were withdrawn. However this work was done independently by us and hence this chapter is included in the thesis. Moreover we have considered here the matrix D(t) whereas Hansen had considered only the constant matrix D. Our treatment is thus somewhat more general than his.

problems. Weak duality theorem, optimality condition, equilibrium conditions and duality theorems are proved. This duality theorem states that if $Z(t) = Z_o(t)$ is a solution to primal problem then a solution $U(t) = Z_o(t)$ and $W(t) = W_o(t)$ exists to the dual problem and the extreme values of two objective functions are equal. Duality theorem for continuous programs differs from the duality theorem given by Dorn [21] , [25] . Here the existence of feasible solutions for respective problems is not sufficient to guarentee the existence of optimizing solutions.

Section II is devoted to the study of duality theorem for continuous convex programming problems. The duality theorem for continuous convex programming problems states that if $Z(t) = Z_o(t)$ is a solution to primal problem then a solution $U(t) = Z_o(t)$ and $W(t) = W_o(t)$ exists to the dual problem and extreme values of two objective functions are equal. Duality theorem for continuous convex programs and finite convex programs differs in the sence that for finite convex programs if both primal and dual have feasible solutions then they have optimal solutions but for continuous the existence of feasible solutions for both programs is not sufficient to guarentee the existence of optimizing solutions.

SECTION - I

Continuous Time Quadratic Programming Problem

Continuous time quadratic programming problem may be stated as:

P.P. (A). Primal program seeks a function $Z(t)$

$$\text{maximizing} \quad \int_0^T \left[a(t)Z(t)+ \tfrac{1}{2}Z'(t)D(t)Z(t)\right]dt = \int_0^T f(t,Z(t))dt \tag{5.1.1}$$

subject to $Z(t)$ is bounded and measurable, (5.1.2)

$$BZ(t) \leq e(t) + \int_0^t CZ(s)ds \quad 0 \leq t \leq T, \tag{5.1.3}$$

$$Z(t) \geq 0 \quad 0 \leq t \leq T \tag{5.1.4}$$

D.P.(A^*) The dual program seeks functions $U(t)$, $W(t)$

minimising $$\int_0^T \left[e(t)W(t) - \tfrac{1}{2}U'(t)D(t)U(t)\right]dt = \int_0^T g(t,U(t),W(t))dt \tag{5.1.5}$$

subject to $W(t)$, $U(t)$ are bounded and measurable (5.1.6)

$$B'W(t) \geq a(t) + D(t)U(t) + \int_t^T C'W(s)ds \quad 0 \leq t \leq T \tag{5.1.7}$$

$$W(t) \geq 0 \quad 0 \leq t \leq T \tag{5.1.8}$$

Functions $Z(t)$ or $U(t)$, $W(t)$ satisfying (5.1.2) (5.1.3) (5.1.4) or (5.1.6) (5.1.7) (5.1.8) respectively are called feasible.

Feasible functions attaining the maximum or minimum value are called optimal.

B, C, $a(t)$, $e(t)$ are same as defined in chapter I and satisfying the same two hypotheses $D(t)$ is an $N \times N$ symmetric negative semi-definite matrix for $0 \leq t \leq T$ such that for all feasible solutions $U(t)$, $D(t)U(t)$ is bounded and measurable. Prime denotes the transpose.

<u>Lemma I</u>. If $(D(t))$ is symmetric negative semi-definite matrix for $0 \leq t \leq T$ then for all vectors $Z(t)$ and $\bar{Z}(t)$

$$\int_0^T \left[\bar{Z}'(t)D(t)\bar{Z}(t) + Z'(t)D(t)Z(t)\right]dt \leq \int_0^T 2\left[\bar{Z}'(t)D(t)Z(t)\right]dt, \quad 0 \leq t \leq T.$$

Proof. From negative semi-definiteness for any $Z(t)$ and $\bar{Z}(t)$

$$\int_0^T (\bar{Z}(t)- Z(t))'D(t)(\bar{Z}(t)- Z(t))dt \leq 0, \qquad 0 \leq t \leq T$$

or

$$\int_0^T \left[\bar{Z}'(t)D(t)\bar{Z}(t)+ Z'(t)D(t)Z(t)\right] dt$$

$$\leq \quad 2\int_0^T \left[\bar{Z}'(t)D(t)Z(t)\right]dt \quad 0 \leq t \leq T.$$

Since $D(t)$ is symmetric.

Theorem (5.1A) (Weak Duality Theorem)

If $Z(t)$ and $U(t)$, $W(t)$ be any feasible solutions then

$$\text{Sup} \int_0^T \left[a(t)Z(t)+ \tfrac{1}{2}Z'(t)D(t)Z(t)\right] dt$$

$$\leq \text{Inf} \int_0^T \left[c(t)W(t)- \tfrac{1}{2}\, U'(t)D(t)U(t)\right] dt.$$

Proof. For feasible $Z(t)$ and $U(t)$, $W(t)$ we have from (5.1.4) and (5.1.7)

$$\int_0^T (Z(t),B'W(t))dt \geqslant \int_0^T \left[a(t)Z(t)+Z'(t)D(t)U(t)\right] dt$$

$$+\int_0^T (Z(t), \int_t^T C'W(s)ds)dt \quad 0 \leq t \leq T \qquad (5.1.9)$$

Similarly from (5.1.8) and (5.1.3) for feasible $Z(t)$ and $W(t)$

$$\int_0^T (W(t),BZ(t))dt \leq \int_0^T (c(t)W(t))dt +\int_0^T (W(t), \int_0^t CZ(s)ds)dt$$

$$0 \leq t \leq T. \qquad (5.1.10)$$

Since $\int_0^T (W(t),BZ(t))dt = \int_0^T (Z(t), B'W(t))dt,$

and $\int_0^T \left(Z(t), \int_t^T C'W(s)ds\right)dt = \int_0^T \left(W(t), \int_0^t CZ(s)ds\right)dt.$

Substracting (5.1.10) from (5.1.9)

$$0 \geqslant \int_0^T \left[a(t)Z(t) + Z'(t)D(t)U(t) - C(t)W(t)\right] dt. \qquad (5.1.11)$$

using lemma I in (5.1.11)

$$0 \geqslant \int_0^T \left[a(t)Z(t) + \tfrac{1}{2}Z'(t)D(t)Z(t) + \tfrac{1}{2}U'(t)D(t)U(t) - c(t)W(t)\right]dt,$$

$$\int_0^T \left[c(t)W(t) - \tfrac{1}{2}U'(t)D(t)U(t)\right]dt \geqslant \int_0^T \left[a(t)Z(t) + \tfrac{1}{2}Z'(t)D(t)Z(t)\right]dt$$

or $\operatorname{Inf} \int_0^T \left[c(t)W(t) - \tfrac{1}{2}U'(t)D(t)U(t)\right] dt \geqslant \operatorname{Sup} \int_0^T \left[a(t)Z(t) + \tfrac{1}{2}Z'(t) D(t)Z(t)\right] dt.$

<u>Theorem (5.1B) (Optimality Condition)</u>

If there exists functions $\bar{Z}(t)$, $\bar{U}(t)$; $\bar{W}(t)$ feasible for their respective problems such that

$$\int_0^T \left[a(t)\bar{Z}(t) + \tfrac{1}{2}\bar{Z}'(t)D(t)\bar{Z}(t)\right] dt = \int_0^T \left[c(t)\bar{W}(t) - \tfrac{1}{2}\bar{U}'(t)D(t)\bar{U}(t)\right] dt \qquad (5.1.12)$$

then $\bar{Z}(t)$, $\bar{U}(t)$; $\bar{W}(t)$ are optimal solutions for their respective problems.

__Proof.__ Let $Z(t)$ be any feasible solution for primal problem and $\overline{U}(t)$, $\overline{W}(t)$ be any feasible solution for dual problem then from weak duality theorem

$$\int_0^T \left[a(t)Z(t)+\tfrac{1}{2}Z'(t)D(t)Z(t)\right] dt \leq \int_0^T \left[c(t)\overline{W}(t)-\tfrac{1}{2}\overline{U}'(t)D(t)\overline{U}(t)\right] dt$$

$$= \int_0^T \left[a(t)\overline{Z}(t)+\tfrac{1}{2}\overline{Z}'(t)D(t)\overline{Z}(t)\right] dt \text{ (from (5.1.12))}$$

or $$\int_0^T \left[a(t)Z(t)+\tfrac{1}{2}Z'(t)D(t)Z(t)\right] dt \leq \int_0^T \left[a(t)\overline{Z}(t)+\tfrac{1}{2}\overline{Z}'(t)D(t)\overline{Z}(t)\right] dt.$$

This shows that $\overline{Z}(t)$ is an optimal solution for primal problem, similarly it can be shown that $\overline{U}(t)$, $\overline{W}(t)$ is an optimal solution for the dual problem.

__Lemma 2.__ If $Z_1(t)$, $Z_0(t)$ are feasible solutions then $Z_2(t) = \lambda Z_0(t) + (1-\lambda) Z_1(t)$, $(0 < \lambda < 1)$ is also feasible.

__Proof.__ $$Z_0(t) \geq 0 \Rightarrow \lambda Z_0(t) \geq 0 \qquad 0 \leq t \leq T. \tag{5.1.13}$$

$$Z_1(t) \geq 0 \Rightarrow (1-\lambda) Z_1(t) \geq 0 \qquad 0 \leq t \leq T. \tag{5.1.14}$$

From (5.1.13) and (5.1.14)

$$Z_2(t) = \lambda Z_0(t) + (1-\lambda) Z_1(t) \geq 0 \qquad 0 \leq t \leq T.$$

$$B\,\lambda Z_0(t) \leq \lambda c(t) + \int_0^t C\,\lambda Z_0(s)ds \qquad 0 \leq t \leq T.$$

$$B(1-\lambda)Z_1(t) \leq (1-\lambda)c(t) + \int_0^t C(1-\lambda)Z_1(s)ds \qquad 0 \leq t \leq T.$$

Therefore

$$B\left[\lambda Z_0(t)+(1-\lambda)Z_1(t)\right] \leq e(t)+\int_0^t C\left[\lambda Z_0(S)+(1-\lambda)\right.$$

$$\left.Z_1(S)\right]dS \qquad 0 \leq t \leq T$$

or $$BZ_2(t) \leq e(t) + \int_0^t C\, Z_2(S)dS \qquad 0 \quad t \quad T.$$

Hence Proved.

Lemma 3. If $Z_0(t)$ is optimal for problem A then it is also optimal for the continuous time linear programming problem L.

L. $$\int_0^T F(t,Z(t))dt = \text{maximize} \int_0^T \left[a(t)Z(t)+Z_0'(t)D(t)Z(t)-\tfrac{1}{2}\right.$$

$$\left.Z_0'(t)D(t)Z_0(t)\right]dt$$

subject to $$BZ(t) \leq e(t)+\int_0^t CZ(S)dS \qquad 0 \leq t \leq T,$$

$$Z(t) \geq 0 \qquad 0 \leq t \leq T,$$

$Z(t)$ is bounded and measurable.

Proof. The constraint sets of A and L are identical. Suppose there exists $Z_1(t)$ satisfying the constraints and such that

$$\int_0^T F(t,Z_1(t))dt > \int_0^T F(t,Z_0(t))\, dt \qquad 0 \leq t \leq T$$

i.e. $$\int_0^T \left[a(t)Z_1(t)+Z_0'(t)D(t)Z_1(t)-\tfrac{1}{2}Z_0'(t)D(t)Z_0(t)-a(t)Z_0(t)\right.$$

$$\left.-Z_0'(t)D(t)Z_0(t)+\tfrac{1}{2}Z_0'(t)D(t)Z_0(t)\right] dt > 0 \qquad 0 \leq t \leq T$$

or $$\int_0^T \left[Z_0'(t)D(t)+a(t)\right]\left[Z_1(t)-Z_0(t)\right] dt > 0 \qquad 0 \leq t \leq T. \quad (5.1.15)$$

Now $Z_2(t) = Z_o(t) + \lambda\left[Z_1(t) - Z_o(t)\right]$ is also feasible for this problem L and

$$\int_0^T \left[a(t)Z_2(t) + \tfrac{1}{2}Z_2'(t)D(t)Z_2(t) - a(t)Z_o(t) - \tfrac{1}{2}Z_o'(t)D(t)Z_o(t)\right] dt$$

$$= \int_0^T \Big\{a(t)\left[Z_o(t) + \lambda\,(Z_1(t) - Z_o(t))\right] + \tfrac{1}{2}\left[Z_o(t) + \lambda(Z_1(t) - Z_o(t))\right]'$$

$$D(t)\left[Z_o(t) + \lambda(Z_1(t) - Z_o(t))\right] - a(t)Z_o(t) - \tfrac{1}{2}Z_o'(t)D(t)Z_o(t)\Big\}\, dt$$

$$= \int_0^T \Big\{\lambda a(t)(Z_1(t) - Z_o(t)) + \tfrac{\lambda}{2}(Z_1(t) - Z_o(t))'D(t)Z_o(t) + \tfrac{\lambda}{2}\, Z_o'(t)$$

$$D(t)(Z_1(t) - Z_o(t)) + \tfrac{\lambda^2}{2}(Z_1(t) - Z_o(t))'D(t)(Z_1(t) - Z_o(t))\Big\}\, dt$$

$$= \lambda\int_0^T \Big\{\left[Z_o'(t)D(t) + a(t)\right]\left[Z_1(t) - Z_o(t)\right] + \tfrac{\lambda}{2}\,(Z_1(t) - Z_o(t))'$$

$$D(t)(Z_1(t) - Z_o(t))\Big\}\, dt. \qquad (5.1.16)$$

Choose λ to be $0 < \lambda < \dfrac{\int_0^T \left[Z_o'(t)D(t) + a(t)\right]\left[Z_1(t) - Z_o(t)\right] dt}{-\tfrac{1}{2}\int_0^T \left[(Z_1(t) - Z_o(t))'D(t)(Z_1(t) - Z_o(t))\right] dt}$

This implies that (5.1.16) is positive, therefore

$$\int_0^T f(t, Z_2(t))\,dt - \int_0^T f(t, Z_o(t))\, dt > 0.$$

This contradicts that $Z_o(t)$ maximizes A. Therefore for all $Z(t)$,

$$\int_0^T F(t,Z(t))dt \leq \int_0^T F(t,Z_o(t)dt.$$

Thus $Z_o(t)$ is the optimal solution to problem L.

<u>Theorem (5.10) Duality Theorem</u>

If $Z(t) = Z_o(t)$ is a solution to problem A then a solution $U(t) = Z_o(t)$; $W(t) = W_o(t)$ exists to problem A* and

$$\int_0^T \left[a(t)Z_o(t)+\tfrac{1}{2}Z_o'(t)D(t)Z_o(t)\right] dt = \int_0^T\left[c(t)W_o(t)- \tfrac{1}{2}Z_o'(t)D(t)Z_o(t)\right] dt.$$

<u>Proof</u>. The dual problem to problem L is

L*. ~~maximize~~ minimize $\displaystyle\int_0^T \left[c(t)W(t)-\tfrac{1}{2}Z_o'(t)D(t)Z_o(t)\right] dt = \int_0^T G(t,W(t))dt.$

(5.1.17)

subject to $\displaystyle B'W(t) \geq a(t)+D(t)Z_o(t)+ \int_t^T C'W(s)ds \qquad 0 \leq t \leq T,$

(5.1.18)

$W(t) \geq 0 \qquad 0 \leq t \leq T$ (5.1.19)

$W(t)$ is bounded and measurable. (5.1.20)

Now from the duality theory for continuous time linear programming problem given by Tyndall [86] .

$$\text{minimum} \int_0^T G(t,W(t))dt = \text{maximum} \int_0^T F(t,Z(t))dt = \int_0^T F(t,Z_o(t))dt.$$

If $W(t) = W_o(t)$ is minimizing the dual problem L* then

$$\int_0^T \left[c(t)W_o(t) - \tfrac{1}{2}Z_o'(t)D(t)Z_o(t)\right]dt = \int_0^T\left[a(t)Z_o(t) + \tfrac{1}{2}Z_o'(t)D(t)Z_o(t)\right]dt.$$

(Since $Z(t) = Z_o(t)$ maximises the primal problem L by lemma III).

$$\text{or} \quad \int_0^T (c(t)W_o(t))dt = \int_0^T \left[a(t)Z_o(t) + Z_o'(t)D(t)Z_o(t)\right] dt. \qquad (5.1.21)$$

Clearly $Z_o(t)$, $W_o(t)$ is feasible for dual problem A^*.

Consider any other feasible solution $U(t)$, $W(t)$ to problem A^* then

$$\int_0^T \left[g(t,Z_o(t),W_o(t)) - g(t,U(t),W(t))\right] dt$$

$$= \int_0^T \left[c(t)W_o(t) - \tfrac{1}{2}Z_o'(t)D(t)Z_o(t) - c(t)W(t) + \tfrac{1}{2}U'(t)D(t)U(t)\right] dt$$

$$= \int_0^T \Big[a(t)Z_o(t) + Z_o'(t)D(t)Z_o(t) - \tfrac{1}{2}Z_o'(t)D(t)Z_o(t) - c(t)W(t)$$

$$+ \tfrac{1}{2} U'(t)D(t)U(t)\Big] dt \quad \text{(using 5.1.21))}$$

$$= \int_0^T \left[a(t)Z_o(t) - c(t)W(t) + \tfrac{1}{2}Z_o'(t)D(t)Z_o(t) + \tfrac{1}{2}U'(t)D(t)U(t)\right] dt$$

$$\leq \int_0^T \left[a(t)Z_o(t) - c(t)W(t) + Z_o'(t)D(t)U(t)\right] dt. \text{ (by lemma I)} \quad (5.1.22)$$

Now from (5.1.3) and (5.1.8)

$$\int_0^T (W(t),BZ_o(t))dt \leq \int_0^T c(t)W(t)dt + \int_0^T (W(t), \int_0^t CZ_o(s)ds)dt,$$

$$(5.1.23)$$

and from (5.1.4) and (5.1.7)

$$\int_0^T (Z_o(t), B'W(t))dt \geqslant \int_0^T \left[a(t)Z_o(t) + Z_o'(t)D(t)U(t)\right] dt$$

$$+ \int_0^T (Z_o(t), \int_t^T C'W(s)ds)dt. \qquad (5.1.24)$$

Substracting (5.1.23) from (5.1.24)

$$0 \geqslant \int_0^T \left[a(t)Z_o(t) + Z_o'(t)D(t)U(t) - c(t)W(t)\right] dt. \qquad (5.1.25)$$

Using (5.1.25) in (5.1.22)

$$\int_0^T \left[g(t,Z_o(t),W_o(t)) - g(t,U(t),W(t))\right] dt \leqslant 0.$$

Therefore $(Z_o(t),W_o(t))$ minimises the dual problem.

Now using (5.1.21)

$$\int_0^T g(t,Z_o(t),W_o(t))dt = \int_0^T \left[c(t)W_o(t) - \tfrac{1}{2}Z_o'(t)D(t)Z_o(t)\right] dt$$

$$= \int_0^T \left[a(t)Z_o(t) + Z_o'(t)D(t)Z_o(t) - \tfrac{1}{2}Z_o'(t)D(t)Z_o(t)\right] dt$$

$$= \int_0^T \left[a(t)Z_o(t) + \tfrac{1}{2} Z_o'(t)D(t)Z_o(t)\right] dt$$

$$= \int_0^T f(t,Z_o(t))dt$$

which shows the equality of objective functions.

Theorem (5.1D) Equilibrium Conditions)

Let $Z(t) = Z_o(t)$ and $U(t) = Z_o(t)$, $W(t) = W_o(t)$ are feasible solutions for primal and dual problems respectively. Then

$$\int_0^T \left[a(t)Z_o(t) + \tfrac{1}{2}Z_o'(t)D(t)Z_o(t)\right] dt = \int_0^T \left[c(t)W_o(t) - \tfrac{1}{2}Z_o'(t)D(t)Z_o(t)\right] dt$$

iff both (i) and (ii) are satisfied for almost all $t \in [0, T]$.

(i) $\quad (BZ_o(t))_i < c_i(t) + \left(\int_0^t CZ_o(s)ds\right)_i$ implies $(W_o(t))_i = 0$,

(ii) $\quad (B'W_o(t))_j > a_j(t) + (D(t)Z_o(t))_j + \left(\int_t^T C'W_o(s)ds\right)_j$ implies $(Z_o(t))_j = 0$.

Proof. Suppose condition (i) and (ii) are satisfied almost everywhere in $[0,T]$, then multiplying jth inequality by $(Z_o(t))_j$, ith inequality by $(W_o(t))_i$ summing over j and i respectively and integrating 0 to T we get

$$\int_0^T (Z_o(t), B'W_o(t))dt = \int_0^T \left[a(t)Z_o(t) + Z_o'(t)D(t)Z_o(t)\right] dt$$

$$+ \int_0^T \left(Z_o(t), \int_t^T C'W_o(s)ds\right)dt,$$

$$\int_0^T (W_o(t), BZ_o(t))dt = \int_0^T (c(t)W_o(t))dt + \int_0^T \left(W_o(t),\right.$$

$$\left.\int_0^t CZ_o(s)ds\right)dt,$$

or $$\int_0^T \left[c(t)W_o(t) - \tfrac{1}{2}Z_o'(t)D(t)Z_o(t)\right] dt = \int_0^T \left[a(t)Z_o(t) + \tfrac{1}{2}Z_o'(t)D(t)Z_o(t)\right] dt.$$

Conversely if the conditions fail for t in some subset of $[0,T]$ having positive measure then we have in the usual way

$$\int_0^T (Z_o(t), B'W_o(t))dt > \int_0^T \left[a(t)Z_o(t) + Z_o'(t)D(t)Z_o(t)\right] dt$$

$$+ \int_0^T (Z_o(t), \int_t^T C'W_o(s)ds)\, dt,$$

$$\int_0^T \left[W_o(t), BZ_o(t)\right] dt < \int_0^T \left[c(t)W_o(t)\right]dt + \int_0^T (W_o(t),$$

$$\int_0^t CZ_o(s)ds)dt$$

or $$\int_0^T \left[a(t)Z_o(t) + \tfrac{1}{2}Z_o'(t)D(t)Z_o(t)\right] dt$$

$$< \int_0^T \left[c(t)W_o(t) - \tfrac{1}{2}Z_o'(t)D(t)Z_o(t)\right] dt.$$

Hence theorem proof.

SECTION - II

Continuous Time Convex Programming Problem

The continuous time convex programming problem is stated as follows:

P.P.(A) Primal program seeks a vector function $Z(t)$

maximizing $\int_0^T f(t, Z(t))dt$ (5.2.1)

subject to $Z(t)$ is bounded and measurable (5.2.2)

$$BZ(t) \leqslant c(t) + \int_0^t CZ(s)ds \qquad 0 \leq t \leq T \qquad (5.2.3)$$

$$Z(t) \geqslant 0 \qquad 0 \leq t \leq T \qquad (5.2.4)$$

Here B,C and c(t) are as defined in Chapter I. f(t,Z(t)) is differentiable concave function of Z(t).

The dual program seeks functions U(t), W(t)

D.P. (A^*)

$$\text{minimizing} \quad \int_0^T \left[f(t,U(t)) - U'(t)\nabla f(t,U(t)) + c'(t)W(t)\right] dt$$

$$= \int_0^T g(t,U(t),W(t))\, dt \qquad (5.2.5)$$

subject to $U(t)$, $W(t)$ bounded and measurable (5.2.6)

$$B'W(t) \geqslant \nabla f(t,U(t)) + \int_t^T C'W(s)ds \qquad 0 \leq t \leq T \quad (5.2.7)$$

$$W(t) \geqslant 0 \qquad 0 \leq t \leq T \quad (5.2.8)$$

where $\nabla f(t,U(t))$ is an n-component vector function consisting partial derivatives of f(t,U(t)) with respect to U(t) and is continuous.

<u>Theorem (5.2A)</u>(<u>Weak Duality Theorem</u>) If Z(t) is feasible for primal program A and (U(t),W(t)) is feasible for dual program A* then

$$\text{Sup} \int_0^T f(t,Z(t))dt \leq \text{Inf} \int_0^T \left[f(t,U(t)) - U'(t)\nabla f(t,U(t)) + C'(t)W(t)\right] dt$$

5.2.3 (5.2.8)

<u>Proof</u>. For feasible Z(t) and W(t) we have from (5.2.4) and (5.2.7)

$$\int_0^T (W(t),BZ(t))dt \leq \int_0^T (c'(t)W(t))dt + \int_0^T (W(t),$$

$$\int_0^t CZ(s)ds)dt \qquad 0 \leq t \leq T, \qquad (5.2.9)$$

5.2.4 5.2.7.

and from (5.2.3) and (5.2.8)

$$\int_0^T (Z(t),B'W(t))dt \geq \int_0^T Z'(t) \nabla f(t,U(t))dt + \int_0^T (Z(t),$$

$$\int_t^T C'W(s)ds)dt \qquad 0 \leq t \leq T. \qquad (5.2.10)$$

Substracting (5.2.9) from (5.2.10)

$$0 \geq \int_0^T Z'(t) \nabla f(t,U(t)) - c'(t)W(t))\, dt \qquad 0 \leq t \leq T$$

or

$$0 \geq \int_0^T \Big[Z'(t) \nabla f(t,U(t)) - U'(t) \nabla f(t,U(t)) + U'(t) \nabla f(t,U(t))$$

$$- c'(t)W(t)\Big]\, dt, \qquad 0 \leq t \leq T. \qquad (5.2.11)$$

Using concavity of f(t,U(t))

$$0 \geq \int_0^T \Big[f(t,Z(t)) - f(t,U(t)) + U'(t) \nabla f(t,U(t)) - c(t)W(t)\Big]\, dt,$$

$$\int_0^T f(t,Z(t))dt \leq \int_0^T \Big[f(t,U(t)) - U'(t) \nabla f(t,U(t))$$

$$+ c(t)W(t)\Big]\, dt,$$

$$\text{Sup} \int_0^T f(t,Z(t))dt \leq \text{Inf} \int_0^T \Big[f(t,U(t)) - U'(t) \nabla f(t,U(t))$$

$$+ c(t)W(t)\Big]\, dt.$$

<u>Theorem (5.2B) (Optimality Condition)</u>. If there exists functions $\bar{Z}(t)$, $\bar{U}(t)$; $\bar{W}(t)$ feasible for their respective problems such that

$$\int_0^T f(t,\bar{Z}(t))dt = \int_0^T \left[f(t,\bar{U}(t)) - \bar{U}'(t) \nabla f(t,\bar{U}(t)) + c(t)\bar{W}(t)\right] dt \tag{5.2.12}$$

then $\bar{Z}(t)$, $\bar{U}(t)$; $\bar{W}(t)$ are optimal solutions for their respective problems.

<u>Proof</u>. Consider any feasible solution $Z(t)$ to primal program and $\bar{U}(t)$, $\bar{W}(t)$ to dual program then from weak duality theorem

$$\int_0^T f(t,Z(t))dt \leq \int_0^T \left[f(t,\bar{U}(t)) - \bar{U}'(t) \nabla f(t,\bar{U}(t)) + c(t)\bar{W}(t)\right] dt$$

$$= \int_0^T f(t,\bar{Z}(t))\, dt \quad (\text{from}(5.2.12))$$

or $\displaystyle\int_0^T f(t,Z(t))dt \leq \int_0^T f(t,\bar{Z}(t))dt$

This shows that $\bar{Z}(t)$ is optimal for primal problem.

Similarly it can be shown that $\bar{U}(t)$, $\bar{W}(t)$ is optimal for dual problem.

<u>Theorem (5.2C)(Equilibrium Conditions)</u> If $Z(t)$ and $U(t) = Z(t)$, $W(t)$ are feasible solutions for primal and dual problems respectively then

$$\int_0^T f(t,Z(t))dt = \int_0^T \left[f(t,Z(t)) - Z'(t) \nabla f(t,Z(t)) + c(t)W(t)\right] dt$$

iff both (i) and (ii) are satisfied for almost all $t \in [0,T]$.

(i) $(BZ(t))_i < e_i(t)+\left(\int_0^t CZ(S)dS\right)_i$ implies $W_i(t) = 0$

for $i = 1,2$ ------ n.

(ii) $(B'W(t))_j > (\nabla f(t,Z(t)))_j + \left(\int_t^T C'W(S)dS\right)_j$ implies $Z_j(t)=0$

for $j = 1,2$ ------ n.

<u>Proof</u>. Suppose conditions (i) and (ii) are satisfied almost everywhere in $[0,T]$ then multiplying ith inequality by $W_i(t)$, jth inequality by $Z_j(t)$ summing over i and j respectively and integrating 0 to T we get

$$\int_0^T (W(t),BZ(t))dt = \int_0^T (e(t)W(t))dt + \int_0^T (W(t),$$

$$\int_0^t C'Z(s)ds)dt \tag{5.1.13}$$

$$\int_0^T (Z(t),B'W(t))dt = \int_0^T \left[Z'(t)\nabla f(t,Z(t))\right]dt + \int_0^T (Z(t),$$

$$\int_t^T C'W(s)ds)dt \tag{5.1.14}$$

or $$\int_0^T \left[Z'(t)\nabla f(t,Z(t))\right]dt = \int_0^T (e(t)W(t))dt.$$

$$\int_0^T f(t,Z(t))dt = \int_0^T \left[f(t,Z(t)) - Z'(t)\nabla f(t,Z(t)) + e(t)W(t)\right] dt.$$

Conversely if the condition fail for t in some subset of $[0,T]$ having positive measure then equality in (5.1.13) and (5.1.14) must be replaced by strict inequality

$$\int_0^T (W(t),BZ(t))dt < \int_0^T c(t)W(t)dt + \int_0^T (W(t), \int_0^t CZ(s)ds)dt,$$

$$\int_0^T (Z(t),B'W(t))dt > \int_0^T \left[Z'(t) \nabla f(t,Z(t))\right] dt + \int_0^T \left[Z(t), \int_t^T C'W(s)ds\right] dt$$

or $$\int_0^T \left[c(t)W(t) - Z'(t) \nabla f(t,Z(t))\right] dt > 0$$

$$\int_0^T f(t,Z(t))dt < \int_0^T \left[f(t,Z(t)) - Z'(t) \nabla f(t,Z(t)) + c(t)W(t)\right] dt$$

Hence theorem proof.

Lemma 1. If $Z(t) = Z_o(t)$ is optimal solution to program A then it is also optimal solution to the linear programming problem L.

P.P.(L) maximize $\int_0^T \left[-f(t,Z_o(t)) + Z'(t) \nabla f(t,Z_o(t))\right] dt = \int_0^T F(t,Z(t))dt$ (5.2.15)

subject to $BZ(t) \leq c(t) + \int_0^t CZ(s)ds \qquad 0 \leq t \leq T,$ (5.2.16)

$Z(t) \geq 0 \qquad 0 \leq t \leq T,$ (5.2.17)

$Z(t)$ bounded and measurable. (5.2.18)

Proof. $Z_o(t)$ is an optimal solution to program A

then

$$\int_0^T f(t,Z_o(t))dt \geq \int_0^T f(t,Z(t))dt. \qquad 0 \leq t \leq T. \qquad (5.2.19)$$

The constraint sets of A and L are identical therefore $Z_o(t)$ is also feasible for the linear programming problem L. Suppose there exists another feasible solution $Z_1(t)$ to problem L with the property that

$$\int_0^T \left[-f(t,Z_o(t))+ Z_1'(t)\nabla f(t,Z_o(t))\right] dt > \int_0^T \left[-f(t,Z_o(t)) + Z_o'(t)\nabla f(t,Z_o(t))\right] dt$$

i.e. $$\int_0^T \left[(Z_1(t)-Z_o(t))'\nabla f(t,Z_o(t))\right] dt > 0 \tag{5.2.20}$$

$Z_o(t)$, $Z_1(t)$ are feasible for program A and L therefore

$Z_2(t) = Z_o(t) + \lambda\,(Z_1(t) - Z_o(t))$, $(0 \leq \lambda \leq 1)$ (5.2.21) is also a feasible solution to both programs A and L.

Using mean value theorem we have

$$\int_0^T \left[f(t,Z_2(t))- f(t,Z_o(t))\right] dt$$

$$= \int_0^T (Z_2(t)-Z_o(t))'\nabla f\left[t,Z_o(t)-\theta\,(Z_2(t) - Z_o(t))\right]dt \qquad 0<\theta<1$$

$$\int_0^T (Z_2(t)-Z_o(t))'\left\{\nabla f\left[t,Z_o(t)-\theta\,(Z_2(t)-Z_o(t))\right]- f(t,Z_o(t))\right\} dt$$

$\nabla f(t,Z_o(t))$

$$+\int_0^T (Z_2(t)-Z_o(t))'\nabla f(t,Z_o(t))\ dt,$$

Using (5.2.21)

$$= \lambda \left(\int_0^T (Z_1(t) - Z_0(t))' \left\{ \nabla f\left[t, Z_0(t) - \theta\lambda\, (Z_1(t) - Z_0(t))\right] - \nabla f(t, Z_0(t)) \right\} dt + \int_0^T \left[(Z_1(t) - Z_0(t))' \nabla f(t, Z_0(t)) \right] dt \right) \qquad (5.2.21)$$

The last term of (5.2.21) is independent of λ and by (5.2.20) it is positive, $\nabla f(t, Z(t))$ is continuous, for λ sufficiently small the first integral may be arbitrarily small. Choosing λ such that

$$\left| \int_0^T (Z_1(t) - Z_0(t))' \left\{ \nabla f\left[t, Z_0(t) - \theta\lambda\, (Z_1(t) - Z_0(t))\right] - \nabla f(t, Z_0(t)) \right\} dt \right| < \left| \int_0^T \left[(Z_1(t) - Z_0(t))' \nabla f(t, Z_0(t)) \right] dt \right|$$

we have $$\int_0^T f(t, Z_2(t))dt > \int_0^T f(t, Z_0(t))dt$$

This contradicts (5.2.19). Therefore $Z_0(t)$ must be the maximizing solution to problem L.

<u>Theorem (5.2D) (Duality Theorem)</u>. If $Z_0(t)$ is an optimal solution to program A then there exists $U(t) = Z_0(t)$, $W(t) = W_0(t)$ such that $(Z_0(t), W_0(t))$ is optimal solution to dual program A^* and the extreme values of two objective functions are equal.

<u>Proof</u>. The dual of the linear programming problem L is

D.P.(L^*_*) minimize $\int_0^T \left[-f(t,Z_o(t)) + c(t)W(t)\right] dt$ (5.2.22)

subject to $B'W(t) \geq \nabla f(t,Z_o(t)) + \int_t^T C'W(s)ds \quad , 0 \leq t \leq T$ (5.2.23)

$W(t) \geq 0 \qquad 0 \leq t \leq T$ (5.2.24)

$W(t)$ bounded and measurable (5.2.25)

Now from the duality theory of continuous linear programming problem given by Tyndall [86]

$$\text{minimum} \int_0^T \left[-f(t,Z_o(t)) + c(t)W(t)\right] dt$$

$$= \text{maximum} \int_0^T \left[-f(t,Z_o(t)) + Z'(t)\nabla f(t,Z_o(t))\right] dt.$$

Since $Z(t) = Z_o(t)$ maximizes the primal problem L by lemma I. If $W(t) = W_o(t)$ minimizes the dual problem L^* then

$$\int_0^T \left[c(t)W_o(t) - f(t,Z_o(t))\right] dt = \int_0^T \left[Z_o'(t)\nabla f(t,Z_o(t)) - f(t,Z_o(t))\right] dt$$

$$\text{or} \int_0^T \left[c(t)W_o(t) - Z_o'(t)\nabla f(t\, Z_o(t))\right] dt = 0 \tag{5.2.26}$$

Clearly $Z_o(t)$, $W_o(t)$ is feasible for dual problem A^*. Consider any other feasible solution $U(t)$, $W(t)$ then

$$\int_0^T \left[g(t,Z_o(t),W_o(t))-g(t,U(t),W(t))\right] dt$$

$$= \int_0^T \left[f(t,Z_o(t))-Z_o'(t)\nabla f(t,Z_o(t))+ c(t)W_o(t)-f(t,U(t))\right.$$

$$\left. + U'(t)\nabla f(t,U(t))- c(t)W(t)\right] dt$$

$$= \int_0^T \left[f(t,Z_o(t))-f(t,U(t))+U'(t)\nabla f(t,U(t))-c(t)W(t)\right] dt$$

Using (5.2.26)

$$\leq \int_0^T \left[(Z_o(t)-U(t))'\nabla f(t,U(t))+U'(t)\nabla f(t,U(t))-c(t)W(t)\right] dt$$

(Using concavity of $f(t,Z(t))$ with respect to $Z(t)$)

$$= \int_0^T \left[Z_o'(t)\nabla f(t,U(t))-c(t)W(t)\right] dt \tag{5.2.27}$$

For feasible $W(t)$ and $Z_o(t)$ we have from (5.2.8) and (5.2.3)

$$\int_0^T (W(t),BZ_o(t))\,dt \leq \int_0^T (c(t)W(t))dt + \int_0^T (W(t),$$

$$\int_0^t CZ_o(s)ds)dt \qquad 0 \leq t \leq T. \tag{5.2.28}$$

Similarly for feasible $Z_o(t)$ and $U(t),W(t)$ we get from (5.2.4) and (5.2.7)

$$\int_0^T (Z_o(t),B'W(t))\,dt \geq \int_0^T Z_o'(t)\nabla f(t,U(t))dt + \int_0^T (Z_o(t),$$

$$\int_t^T C'W(s)ds)dt \qquad 0 \leq t \leq T \tag{5.2.29}$$

Substracting (5.2.28) from (5.2.29)

$$0 \geq \int_0^T \left[-c(t)W(t)+Z_o'(t)\nabla f(t,U(t))\right] dt \qquad (5.2.30)$$

Using (5.2.30) in (5.2.27)

$$\int_0^T \left[g(t,Z_o(t),W_o(t)- g(t,U(t),W(t))\right] dt \leq 0.$$

Therefore $(Z_o(t), W_o(t))$ minimizes the dual problem.

Now using (5.2.26) we get

$$\int_0^T f(t,Z_o(t))dt = \int_0^T \{f(t,Z_o(t))- Z_o'(t) \nabla f(t,Z_o(t))+c(t)W_o(t)\}dt$$

This shows the equality of objective functions.

BIBLIOGRAPHY

1. Abadie, J. (ed) (1967) "Nonlinear Programming" North-Holland Publishing Company, Amsterdam 1967.

2. Ahlfors, L.V. (1953) Complex Analysis New York: McGraw-Hill 1953.

3. Apostol, T.M. "Mathematical Analysis" A Modern Approach to Advance Calculus. Addison-Wesley Publishing Company. Inc. Reading Massachusetts, U.S.A.

4. Arrow, K.J.; Hurwicz, L. and Uzawa, H. (1961) "Constraint Qualifications in Maximization Problems" Naval Res. Logist. Quart. 8 (1961) 175-191.

5. Arrow, K.J.; Hurwicz, L. and Uzawa, H. (1958) "Studies in Linear and Nonlinear Programming" Stanford, California; Stanford University Press. 1958.

6. Arrow, K.J.; Hurwicz, L. (1951) "Reduction of Constrained Maxima to Saddle-Point Problems in [72] 1-20.

7. Bellman, R. (1957) "Dynamic Programming" Princeton University Press, Princeton 1957.

8. Berge, C. and Ghouila-Houri, A. (1965) "Programming Games and Transportation Networks" John Wiley and Sons Inc. New York 1965.

9. Boot, J.C.G. (1964) "Quadratic Programming" North-Holland Publishing Company, Amsterdam (1964).

10. Bratton, D. (1955) "The Duality Theorem in Linear Programming" Cowles Commission Discussion Paper Math. No. 427 (Cowles Foundation Yale University 1955).

11. Charnes, A.; Cooper, W.W.; & Kortanek, K. (1962) "A Duality Theory for Convex Programs with Convex Constraints" Proc. Am. Math. Soc. Vol. 68 (1962) 605-608.

12. Charnes, A.; Cooper, W.W.; & Kortanek, K. (1965) "On Representations of Semi-Infinite Programs which have no Duality Gaps." Management Science Vol. 12. No. 1 Sept. 1965. 113-121.

13. Courant, R. (1953) "Differential and Integral Calculus" 2nd ed. Vol. 2 Interscience Publishers Inc. New York. 1953.

14. Cottle, R.W. (1963) "Symmetric Dual Quadratic Programs" Quaterly of Applied Mathematics Vol. 12. 1963.

15. Cottle, R.W. (1964) "Note on a Fundamental Theorem in Quadratic Programming" J. Soc. Ind. and App. Maths. Vol. 12, No. 3, 1964.

16. Cottle, R.W. (1964) "Nonlinear Programs with Positively Bounded Jacobians" ORC 64-12 (RR) 22 June, 1964 University of California-Berkeley.

17. Dantzig, G.B. & Orden, A. (1953) "Duality Theorems" RM 1265 The rand Corporation Oct. 1953.

18. Dantzig, G.B.; Eisenberg, E. & Cottle, R.W. (1965) "Symmetric Dual Nonlinear Programs" Paci. Jour. of Math. 1965. 809-812.

19. Dantzig, G.B. (1963) "Linear Programming and Extensions" Princeton University Press Princeton New Jersey 1963.

20. Dennis, J.B. (1959) "Mathematical Programming and Electrical Networks" John Wiley and Sons Inc. New York 1959.

21. Dorn, W.S. (1960) "Duality in Quadratic Programming" Quart. Appl. Math. 18(1960) 155-162.

22. Dorn, W.S. (1960) "A Symmetric Dual Theorem for Quadratic Programming" J. Operations Res. Soc. Japan 2(1960) 93-97.

23. Dorn, W.S. (1961) "On Lagrange Multipliers and Inequalities" Operations Research Vol. 9 No. 1 (1961).

24. Dorn, W.S. (1961) "Self Dual Quadratic Programs" J. Soc. Indust. Appl. Math. 9(1961) 51-54.

25.	Dorn, W.S.	(1960)	"A Duality Theorem for Convex Programs" I.B.M. Journal of Research and Development 4. (1960) 407-413.
26.	Duffin, R.J.	(1956)	"Infinite Programs" Linear Inequalities and Related System. H.W. Kuhn, A.W. Tucker Princeton (1956).
27.	Duffin, R.J. & Karlovitz, L.A.	(1965)	"An Infinite Linear Program with Duality Gap" Management Science Vol. 12. No. 1, 1965.
28.	Duffin, R.J.; Peterson, E.L.	(1966)	"Duality Theory for Geometric Programming" Siam Jour. of App. Math. vol. 14, No. 6, 1966.
29.	Eisenberg, E.	(1961)	"Duality in Homogeneous Programming" Proc. Amer. Math. Soc. 12. 783-787, (1961).
30.	Eisenberg, E.	(1961)	"A Note on Semi Definite Matrices" Research Report No. 9 Operations Research Centre. University of California, Berkeley (1961).
31.	Fan, K.	(1965)	"A Generalization of the Alaogul-Bourbok Theorem and its Applications" Math. 2. 88 (1965) 48-60
32.	Fan, K.	(1952)	"Fixed Point and the Minimax Theorem in Locally Convex Topological Linear Spaces" Proc. Natu. Acad. Sciences. U.S. 38 (1952) 121-126.
33.	Fenchel, W.	(1953)	"Convex Cones Sets and Functions" (Mimeographed) Princeton 1953.
34.	Gale, D.	(1960)	"The Theory of Economic Models" McGraw Hill Book Company New York (1960)
35.	Gupta, R.P.	(1968)	"Symmetric Dual Quadratic Program in Complex Space" Accepted for Publication in "Proceedings of the Indian Academy of Sciences".
36.	Graves, R.L. & Wolfe, P. (ed)	(1963)	"Recent Advances in Mathematical Programming" McGraw-Hill Book Company Inc. (1963).
37.	Hadley, G.	(1961)	"Linear Algebra" Reading Mass Addison Wesley 1961.

38. Hadley, G. (1962) "Linear Programming" Reading Mass: Addison Wesley Publishing Company 1962.

39. Hadley, G. (1964) "Nonlinear and Dynamic Programming" Reading Mass: Addison Wesley 1964.

40. Hanson, M.A. (1961) "A Duality Theorem for Nonlinear Programming with Nonlinear Constraints" Astra. Jour. of Stats. Vol. 3, 64-72, (1961).

41. Hanson, M.A. (1964) "Duality and Self Duality in Mathematical Programming" Jour. Soc. Indu. Appl. Maths. Vol. 12, No. 2, 1964.

42. Hanson, M.A. & Mond, B. (1967) "Quadratic Programming in Complex Space" Jour. of Math. Anal. and Applications Vol. 20 No. 3, (1967).

43. Hanson, M.A. (1967) "A Continuous Leontief Production Model with Quadratic Objective Function" Econometrica Vol. 35 No. 3-4 July-Oct. 1967.

44. Hanson, M.A. (1968) "Duality for a Class of Infinite Programming Problem " SIAM. Jour Math. Vol. 16 No. 2 (1968).

45. Hanson, M.A.; Mond, B. (1968) "A Class of continuous Convex Programming Problems" Jour. Math. Analy. Appl. Vol. 22 427-437 (1968).

46. Huard, P. (1962) "Dual Programs" I.B.M. Jour. Res. Dev. Vol. 6 (1962) pp. 137-139.

47. John, F. (1948) "Extremum Problems with Inequalities as side Conditions" Studies and Essays Courant Anniversary Volume K.O. Friendricks, O.E. Neugebauer and J.J. Stocker (eds) Interscience Publishing Inc. New York (1948) 187-204.

48. Karlin, S. (1959) "Mathematical Methods and Theory in Games Programming and Economics Vol. 1 Addison-Wesley Publishing Company Inc. Reading Massachusetts (1959).

49. Kanti Swarup (1965) "A Duality Theorem for Nonlinear Programming" ZAMM(Germany) Vol. 45 (1965) pp. 311-314.

50. Kakutani, S. (1941) "A Generalization of Brouwer's fixed point Theorem" Duke Math. 7 8(1941) 457-459.

51. Koopmans, T.C.(ed) (1951) "Activity Analysis of Production and Allocation" John Wiley and Sons Inc. New York (1951).

52. Kreindler, E. (1966) "Reciprocal Optimal Control Problem" Jour. Math. Analy. Appli. Vol. 14 (1966) 141-152.

53. Kretchmer, K.S. (1961) "Programmes in Paired Spaces" Canadian Jour. Math. Vol. 13 (1961) pp. 221-238.

54. Kuhn, H.W.; and Tucker, A.W. (1951) "Nonlinear Programming" in Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability 1951 pp. 481-492.

55. Kuhn, H.W. and Tucker, A.W.(eds) (1956) "Linear Inequalities and Related Systems" Annals of Mathematics Studies no. 38 Princeton University Press Princeton New Jersey 1956.

56. Kumar, T.K. (1967) "A Duality Theorem for Continuous-time Linear Programming Problem" Unternehmensforschung Vol. 10 No. 4 (1967) 224-236.

57. Lehman, R.S. (1954) "On the Continuous Simplex Method" Rand Research Memorandum RM-1386, Santa Monica California 1954.

58. Lemke, C.E. (1954) "The Dual Method of Solving the Linear Programming Problem" Nav. Res. Log. Quart. Vol. 1, 1954 (36-47).

59. Levinson, N. (1966) "Linear Programming in Complex Space" Jour. of Math. Anly. Appli. Vol. 14 (1966) 44-62.

60. Levinson, N. (1966) "A Class of Continuous Linear Programming Problems". Journal of Mathematical Analysis and its Applications Vol. 16, No. 1, (1966).

61. Mangasarian, O.L. (1962) "Duality in Nonlinear Programming" Quart. App. Maths. Vol. 20, (1962) 300-302.

62. Mangasarian, O.L. (1965) "Pseudo Convex Functions" Journal on SIAM Control Vol. 3 No. 2 (1965).

63. Mangasarian, O.L. & Ponstein, J. (1965) "Minimax and Duality in Nonlinear Programming" Journal of Mathematical Analysis and its Application Vol. 11 (1965) 504-518.

64. Mehndiratta, S.L. (1966) "General Symmetric Dual Programs" Operations Research, Vol. 14 No. 1, (1966).

65. Mehndiratta, S.L. (1967) "Symmetry and Self Duality in Nonlinear Programming" Numerisch Mathematik, Vol. 10 pp. 103-109 (1967).

66. Mond, B. (1965) "A Symmetric Dual Theorem for Nonlinear Programming" Quart. Appl. Maths. Vol. 23 (1965) pp. 265-269.

67. Mond, B. & Cottle, R.W. (1966) "Self Duality in Mathematical Programming" Journal. SIAM. App. Maths. Vol. 14 No. 3 (1966).

68. Mond, B. & Hanson, M.A. (1967) "Duality in Variational Problem" Jour. Math. Analy. Appli. Vol. 18 (1967).

69. Mond, B. & Hanson, M.A. (1968) "Symmetric Duality for Variational Problem " Jour. Math. Analy. Appli. Vol. 23 No. 1 (1968).

70. Mond, B. & Hanson, M.A. (1968) "Duality for Control Problem" SIAM Journal Control Vol. 6 No. 1 (1968).

71. Neumann, J.Von. (1947) "On a Maximization Problem" Institute for Advanced Study, Princeton New Jersey November 1947.

72. Neyman, J.(ed) (1951) "Proceeding of the Third Berkeley Symposium on Mathematical Statistics and Probability" University of California Press, Berkeley and Los Angeles. 1951.

73. Neymann, J.(eds)(1951) "Proceeding of the Second Berkeley Symposium on Mathematical Statistics and Probability" University of California Press, Berkeley and Los Angeles. 1951.

74. Ostrowski, A.M. (1951) & Taursky, O. "On the Variation of the Determinant of a Positive Definite Matrix" Nederl, Akad. Wetensch. Proc. Ser. A54 (1951) 383-385

75. Pallu, R. & (1966) Barriere, DeLa "Duality in Dynamic Optimization" J. SIAM Control Vol. 4 No. 1 (1966)

76. Pearson, J.D. (1965) "Reciprocity and Duality in Control Programming Problem " Jour. Math. Analy. Appli. Vol. 10 (1965) pp. 388-408.

77. Pearson, J.D. (1966) "Duality and Decomposition Technique" J. SIAM. Control Vol. 4, No. 1 (1966) pp. 164-172.

78. Poinstein, J. (1967) "Seven Kinds of Convexity" SIAM Review Vol. 9, No. 1, 1967.

79. Rissanen, J. (1967) "Duality without Convexity" Jour. Math. Analy. Appl. Vol.18, 269-275 (1967).

80. Sion, M. (1958) "On General Minimax Theorem" Pacific Journal Math. Vol. 8 (1958).

81. Sreedharan, V.P.(1965) "A Short Proof of the Duality Theorem of Linear Programming" J. Soc. Indust. Appl. Math. Vol. 13, No. 2, (1965).

82. Sinha, S.M. (1966) "A Duality Theory for Nonlinear Programming" Management Science Vol. 12, No. 5 (1966).

83. Tucker, A.W. (1960) "A Combinatorial Equivalence of Matrices" in Combinatorial Analysis Proceedings of Symposia in Applied Mathematics 10 American Mathematical Society 1960.

84. Tucker, A.W. (1957) "Linear and Nonlinear Programming" Operations Research Vol. 5, No. 2 1957, pp. 244-257.

85. Tyndall, W.F. (1965) "A Duality Theorem for a Class of Continuous Linear Programming Problems" Jour. Soc. Indust. App. Math. Vol. 13 (1965) pp. 644-666.

86. Tyndall, W.F. (1967) "An Extended Duality Theorem for Continuous Linear Programming Problems" SIAM. Jour. Appli. Maths. Vol. 15, No. 5 (1967) pp. 1294-1298.

87. Valentine, F.A. (1937) The Problem of Lagrange with Differential inequalities as added side conditions contribution to the calculus of variation university of Chicago Press 1937 407-448.

88. Vajda, S. (1961) "Mathematical Programming" Reading Marsch. Addison Wesley (1961).

89. Wolfe, P. (1961) "A Duality Theorem for Nonlinear Programming" Quart. Appl. Math. Vol. 19 (1961), pp. 239-244.

90. Wolfe, P. "A Dynamic Linear Programming Model" Privately Circulated.